Numerical solutions of fuzzy partial differential equation and its application in computational mechanics

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Numerical example

Plane stress problem
in theory of elasticity

Plane stress problem in theory of elasticity

$$\frac{E}{2(1+\nu)} u_{\alpha,\beta\beta} + \frac{E}{2(1-\nu)} u_{\beta,\beta\alpha} + \rho f_\alpha = 0, \quad \alpha,\beta = 1,2$$

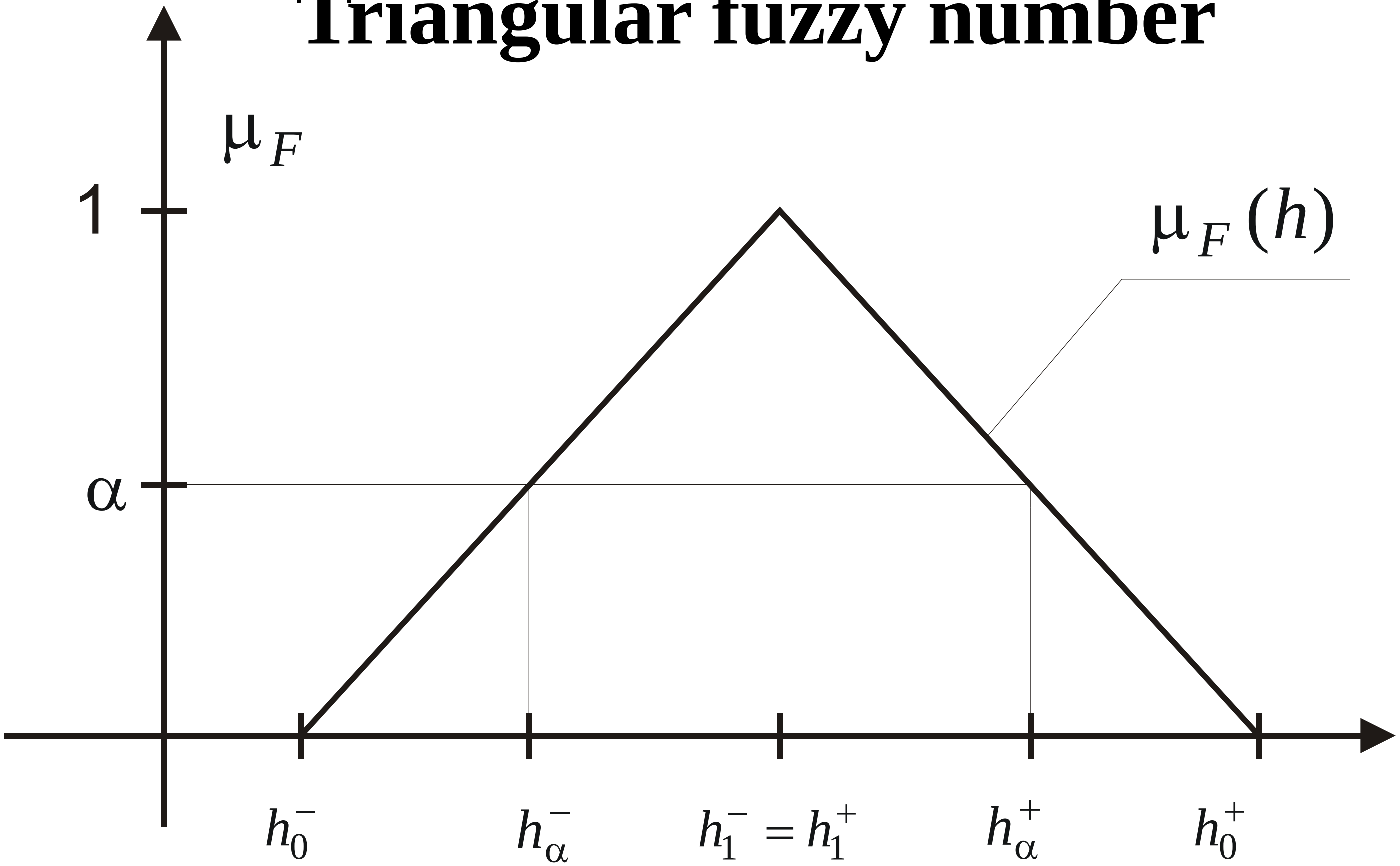
$$u_\alpha = \overset{*}{u}_\alpha, \quad x \in \partial\Omega_u$$

$$\sigma_{\alpha\beta} n_\beta = \overset{*}{t}_\alpha, \quad x \in \partial\Omega_\sigma$$

ρ - mass density,
E,ν - material constant,
f_α - mass force.

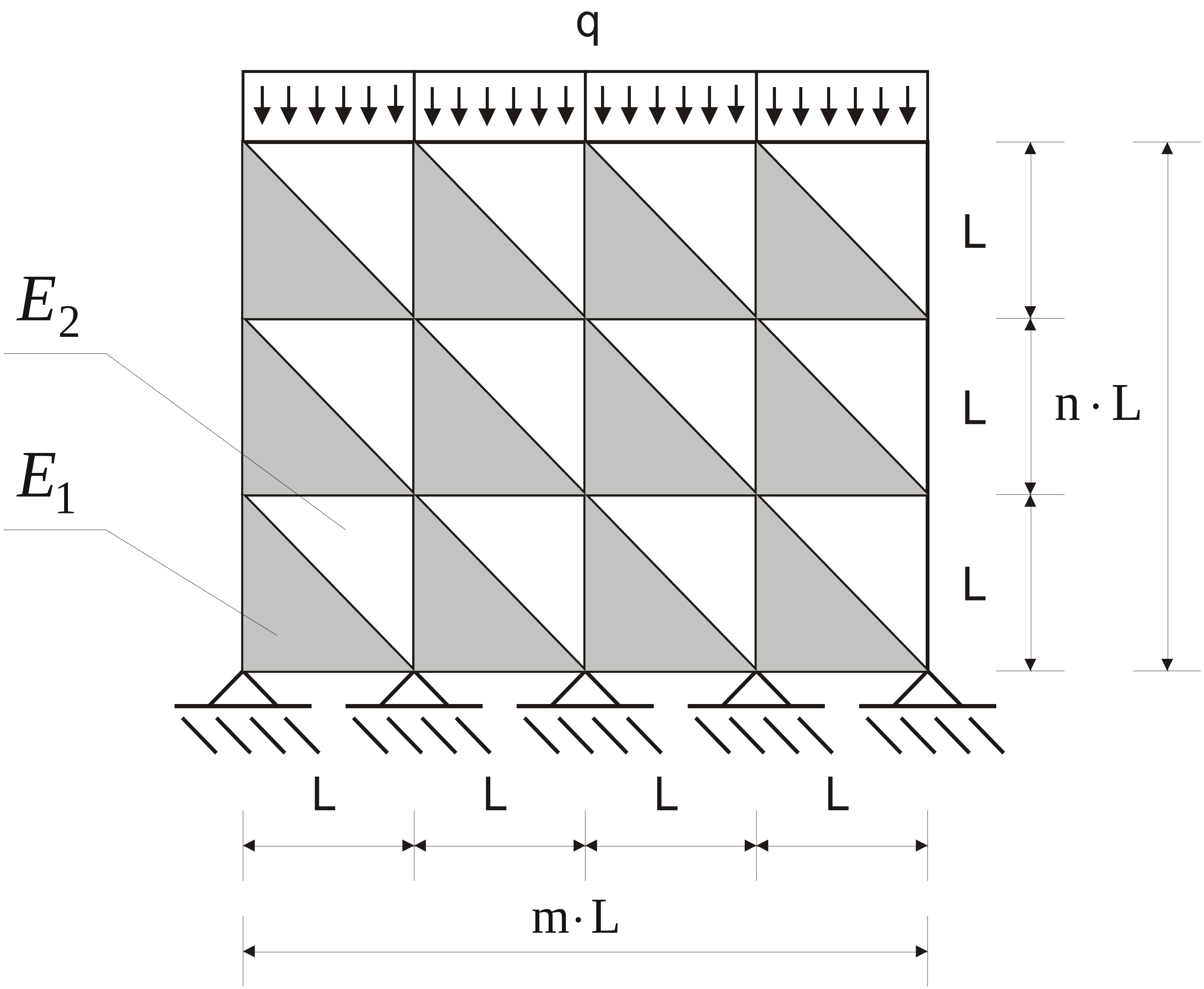
$$u_{\alpha,\beta} = \frac{\partial u_\alpha}{\partial x_\beta}$$

Triangular fuzzy number



$$\hat{h}_\alpha = \{h : \mu_F(h) \geq \alpha\}$$

$$\hat{h}_\alpha = [h_\alpha^-, h_\alpha^+]$$


q
L
L
L
n · L
E2
E1
L
L
L
L
m · L

Data

$$\hat{E}_0^1 = [189, 231], [GPa], \ \alpha = 0,$$

$$\hat{E}_1^1 = [210, 210], [GPa], \ \alpha = 1,$$

$$\hat{E}_0^2 = [189, 231], [GPa], \ \alpha = 0,$$

$$\hat{E}_1^2 = [210, 210], [GPa], \ \alpha = 1.$$

$$\nu = 0.3 \quad q = 1\left[\frac{kN}{m}\right], \quad L = 1[m]$$

Time of calculation

n	m	DOF	Elements	Time
5	5	72	50	00:00:01
10	10	242	200	00:00:09
20	20	882	800	00:03:50
30	40	2542	2400	01:27:52

Processor: AMD Duron 750 MHz

RAM: 256 MB

Numerical example

Shell structure with fuzzy material properties

Equilibrium equations of shell structures

$$T^{\beta\alpha}|_{\beta} - b^{\alpha}_{\gamma} M^{\beta\gamma}|_{\beta} + b^{\alpha} = 0$$

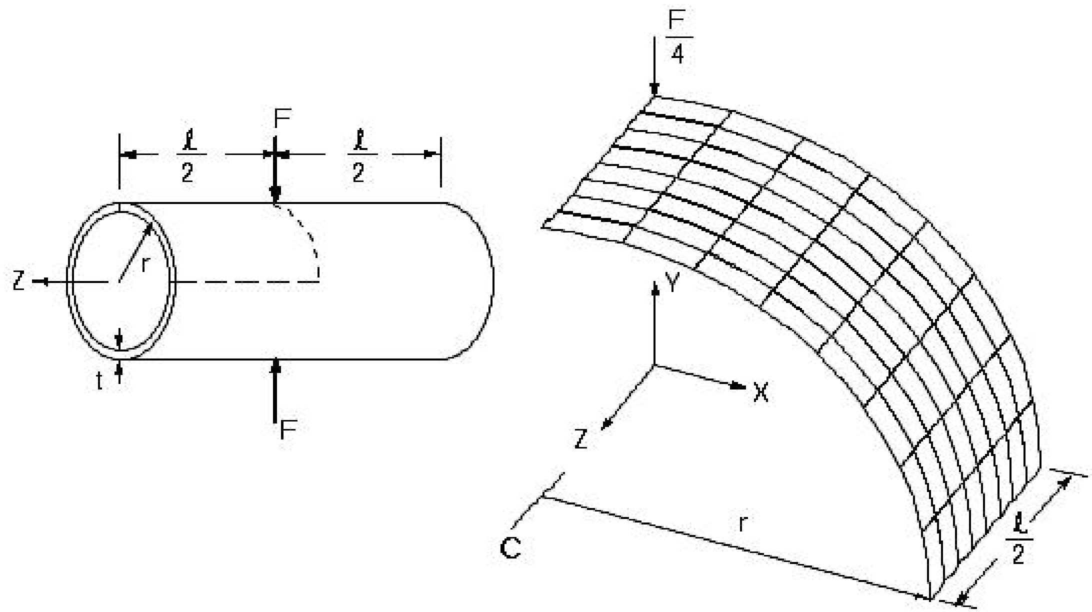
$$T^{\beta\alpha} b_{\alpha\beta} + M^{\alpha\beta}|_{\beta\beta} + b^{3} = 0$$

$$T^{\beta\alpha} n_{\beta} + b^{\alpha}_{\gamma} M^{\beta\gamma} n_{\beta} = p^{\alpha}, \quad x \in \partial\Omega$$

$$M^{\beta\alpha}|_{\beta}\, n_{\alpha} + \frac{d}{ds}(M^{\alpha\beta}\tau_{\alpha} n_{\beta}) = p^{3}, \quad x \in \partial\Omega$$

where

$$u^{\alpha}|_{\beta} = u^{\alpha}{}_{,\beta} + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} u^{\gamma}, \qquad u^{\alpha}{}_{,\beta} = \frac{\partial u^{\alpha}}{\partial x^{\beta}}, \quad \alpha, \beta = 1,2$$



Numerical data (α=0)

$$E \in [2.0 \cdot 10^5, 2.2 \cdot 10^5][MPa], \quad \nu \in [0.2, 0.3],$$

L=0.263 [m], r=0.126 [m], F=444.8 [N], t= $2.38 \cdot 10^{-3}$ [m]

Numerical results (fuzzy displacement)

α=0: $u \in \left[-0.043514, -0.03748\right]$ [m]

α=1: u = -0.04102 [m].

Using this method we can obtain the fuzzy solution in one point.

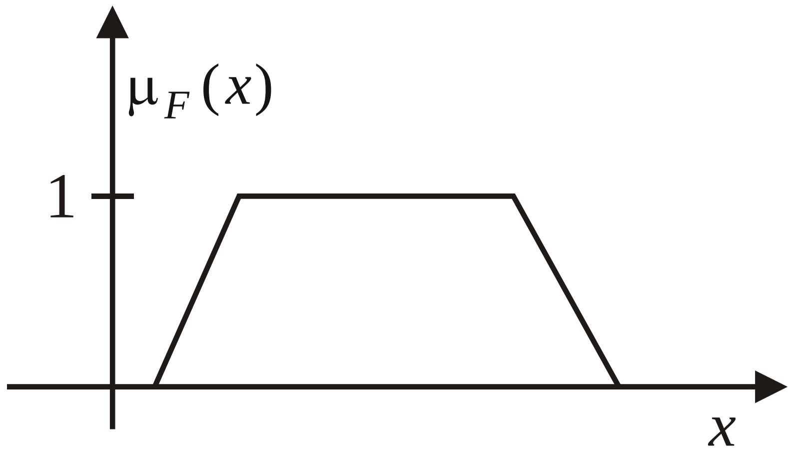
The solution was calculated by using the ANSYS FEM program.

The main goal of this presentation is to describe methods of solution of partial differential equations with fuzzy parameters.

Basic properties of fuzzy sets

Fuzzy sets

$$\mu_F : R \ni x \to \mu_F(x) \in [0,1] \subset R$$



$$\mu_{A\cap B}(x) = min\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$$

$$\mu_{A\cup B}(x) = max\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A\cup B}(x) = S(\mu_A(x), \mu_B(x))$$

Extension principle

$$y = f(x_1, x_2, ..., x_n)$$

$$\mu_{f(F)}(y) = \max_{y=f(x_1,x_2,...,x_n)} \min\{\mu_F(x_1), \mu_F(x_2), ..., \mu_F(x_n)\}$$

$$\mu_{f(F)}(y) = \max_{y=f(\mathbf{x})} \mu_F(\mathbf{x}),$$

$$f : R^n \to R, \qquad F \in F(R^n), \qquad f(F) \in F(R).$$

Fuzzy equations

Fuzzy algebraic equations

$$\mathbf{H}(\mathbf{y},\mathbf{h}) = \mathbf{0} \qquad \Leftrightarrow \qquad \mathbf{y} = \mathbf{y}(\mathbf{h})$$

$$\mathbf{H}: R^n \times R^m \to R^n, \qquad F \in F(R^m)$$

$$\mu_F : R^m \ni \mathbf{h} \to \mu_F(\mathbf{h}) \in [0,1]$$

$$\mu_{H(F)}(\mathbf{y}) = \max_{\mathbf{h}:\mathbf{H}(\mathbf{y},\mathbf{h})=\mathbf{0}} \mu_F(\mathbf{h})$$

Fuzzy differential equation (example)

$$\frac{dy}{dx} = h \cdot x, \qquad y(0) = y_0, \qquad h \in F \in F(R)$$

$$y(x,h) = \frac{hx^2}{2} + y_0$$

$$\mu(\xi \mid y_F(x)) = \max_{h:\ \xi = \frac{h \cdot x^2}{2} + y_0} \mu_F(h)$$

$$y_F(x) \in F(R)$$

Definition of the solution of fuzzy differential equation

$$\frac{dy}{dx} = f(x, y, h), \qquad y(0) = y_0, \quad h \in F \in F(R)$$

$$\mu(\xi \mid y_F(x)) = \max_{h:\ \xi = y(x,h),\ \frac{dy}{dx} = y(x,h),\ y(0) = y_0} \mu_F(h)$$

$$y_F(x) \in F(R)$$

Fuzzy partial differential equations

$$\mathbf{H}\left(\mathbf{x},\mathbf{u},\frac{\partial \mathbf{u}}{\partial \mathbf{x}},\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2},\ldots,\frac{\partial^k \mathbf{u}}{\partial \mathbf{x}^k},\mathbf{h}\right)=\mathbf{0}, \quad \mathbf{u}\in V, \quad \mathbf{h}\in F\in F(R^m)$$

$$\mu(\boldsymbol{\xi}\,|\,\mathbf{u}_F(\mathbf{x})) = \max_{\mathbf{h}:\ \boldsymbol{\xi}=\mathbf{u}(\mathbf{x},\mathbf{h}),\ \mathbf{H}\left(\mathbf{x},\mathbf{u},\mathbf{h},\frac{\partial \mathbf{u}}{\partial \mathbf{x}},\ldots,\frac{\partial^k \mathbf{u}}{\partial \mathbf{x}^k},\mathbf{h}\right)=\mathbf{0},\ \mathbf{u}\in V} \mu_F(\mathbf{h})$$

$$\mathbf{u}_F(\mathbf{x})\in F(R^n)$$

$$[1,2]\cdot x=[2,8]$$

Algebraic solution set

$$[1,2]\cdot[x^-,x^+]=[2,8]$$

$$\hat{x}=[2,4]$$

United solution set

$$\hat{x}=\{x:\exists a\in[1,2],\exists b\in[2,4],a\cdot x=b\}=[1,4]$$

Controllable solution set

$$\hat{x}=\{x:\exists a\in[1,2],\forall b\in[2,8],a\cdot x=b\}$$

$$\hat{x}=\{x:[1,2]\cdot x\supseteq[2,8]\}=\varnothing$$

Tolerable solution set

$$\hat{x}=\{x:\forall a\in[1,2],\exists b\in[2,8],a\cdot x=b\}$$

$$\hat{x}=\{x:[1,2]\cdot x\subseteq[2,8]\}=[2,4]$$

Remarks

Buckley J.J., Feuring T., Fuzzy differential equations. Fuzzy Sets and System, Vol.110, 2000, 43-54

$$F_\alpha(x) = cl\{y : \mu_{F(x)}(y) \geq \alpha\}$$

$$F_\alpha^-(x) = inf\ F_\alpha(x), \quad F_\alpha^+(x) = sup\, F_\alpha(x).$$

$$\frac{d}{dx} F_\alpha(x) = \frac{d}{dx}[F_\alpha^-(x), F_\alpha^+(x)] = \left[\frac{dF_\alpha^-(x)}{dx}, \frac{dF_\alpha^+(x)}{dx} \right] = \frac{dF_\alpha(x)}{dx}$$

- Goetschel-Voxman derivative,
- Seikkala derivative,
- Dubois-Prade derivative,
- Puri-Ralescu derivative,
- Kandel-Friedman-Ming derivative,
- etc.

This derivative leads to another definition of the solution of the fuzzy differential equation.

Applications of fuzzy equations in computational mechanics

Physical interpretations of fuzzy sets

Equilibrium equations of isotropic linear elastic materials

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \qquad \sigma_{ij} = C_{ijkl} \varepsilon_{kl},$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

$$u_i = u_i^*, \quad x \in \partial \Omega_u, \qquad \sigma_{ij} n_j = t_i^*, \quad x \in \partial \Omega_\sigma,$$

$$u(x,t)\big|_{t=0} = u^*(x), \quad x \in \Omega.$$

Uncertain parameters

- Fuzzy loads,

- Fuzzy geometry,

- Fuzzy material properties,

- Fuzzy boundary conditions e.t.c.

Modeling of uncertainty

Probabilistic methods
$X_\Omega : \Omega \to R, \quad f_X(x).$
Semi-probabilistic methods $x \quad \to \quad \gamma \cdot \mathrm{x}$

Usually we don't have enough information to calculate probabilistic characteristics of the structure.

We need another methods of modeling of uncertainty.

Random sets interpretation of fuzzy sets

$$\hat{H}_\Omega : \Omega \ni \omega \rightarrow \hat{H}_\Omega(\omega) \in I(R)$$

$$Pl(A) = P_\Omega\{\omega : \hat{H}_\Omega(\omega) \cap A \neq \emptyset\}$$

$$\hat{H}_\Omega(\omega_1) \supseteq \hat{H}_\Omega(\omega_2) \supseteq \ldots \supseteq \hat{H}_\Omega(\omega_n)$$

$$\mu_F(h) = Pl(\{h\}) = P_\Omega\{\omega : h \in \hat{H}_\Omega(\omega)\}$$

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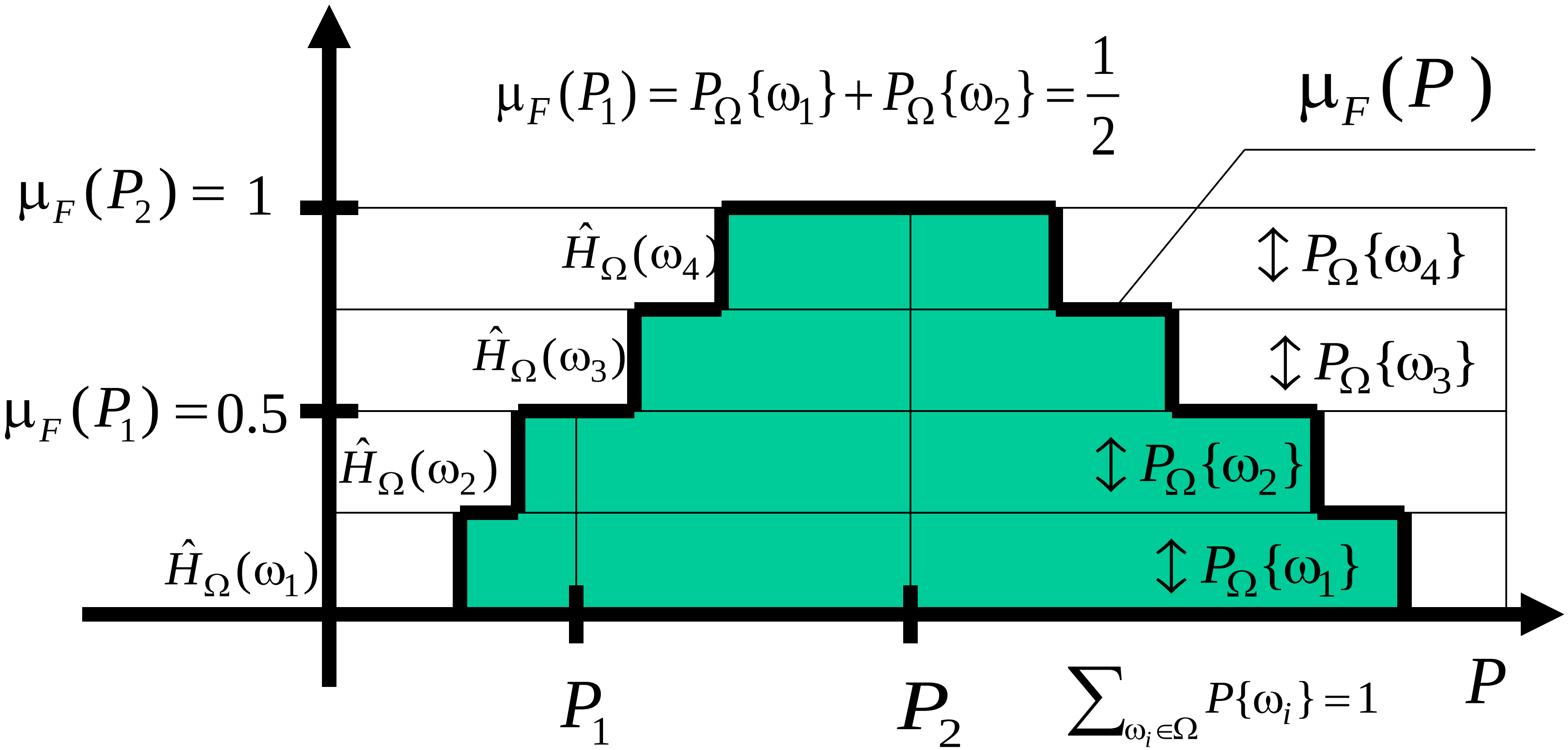
Random sets interpretation of fuzzy sets



This is not
a probability density function
or a conditional probability
and
cannot be converted
to them.

$$\hat{X}_\Omega : \Omega \ni \omega \rightarrow \hat{X}_\Omega(\omega) \in I(R)$$

$$X_\Omega : \Omega \ni \omega \rightarrow X_\Omega(\omega) \in R$$

$$\forall \omega \in \Omega,\ \ X_\Omega(\omega) \in \hat{X}_\Omega(\omega)$$

$$P_\Omega^{X_\Omega}([a,b]) = P_\Omega\{\omega : X_\Omega(\omega) \in [a,b]\}$$

$$Pl([a,b]) = P_\Omega\{\omega : \hat{X}_\Omega(\omega) \cap [a,b] \neq \emptyset\}$$

$$P_\Omega^{X_\Omega}([a,b]) \leq Pl([a,b])$$

Random sets

$$\hat{H}_\Omega : \Omega \ni \omega \to \hat{H}_\Omega(\omega) \in I(R)$$

Probabilistic methods

$$H^+_\Omega(\omega) = H^-_\Omega(\omega) = H_\Omega(\omega)$$

Fuzzy methods

$$\hat{H}_\Omega(\omega_1) \supseteq \hat{H}_\Omega(\omega_2) \supseteq ... \supseteq \hat{H}_\Omega(\omega_n)$$

Semi-probabilistic methods (interval methods)

$$\hat{H}_\Omega(\omega_1) = \hat{H}_\Omega(\omega_2) = ... = \hat{H}_\Omega(\omega_n)$$

or

another procedures.

Design of structures with fuzzy parameters

$$P_f = Pl\{g(\mathbf{h}) < 0\} \leq P_f^0$$

$$P_f = \sup_{h:\, g(h)<0} \mu_F(h)$$

Equation with fuzzy and random parameters

$$X_{\Omega} : \Omega \ni \omega \to X_{\Omega}(\omega) \in R,$$

$$\hat{H}_{\Gamma} : \Gamma \ni \gamma \to \hat{H}_{\Gamma}(\gamma) \in I(R), \quad \mu_F(h) = P_{\Gamma}\{\gamma : h \in \hat{H}_{\Gamma}(\gamma)\}.$$

$$P_f^+ = P_{\Omega\times\Gamma}\{(\omega,\gamma) : g(X_{\Omega}(\omega), \hat{H}_{\Gamma}(\gamma)) < 0\}$$

$$P_f^+ = P_{\Omega\times\Gamma}\{(\omega,\gamma) : g(X_\Omega(\omega), \hat{H}_\Gamma(\gamma)) < 0\}$$

$$\mu_{x(F)}(x) = \sup_{h:g(x,h)<0} \mu_F(h)$$

$$P_f^+ = \sum_x P_\Omega\{x\} \cdot \mu_{x(F)}(x)$$

$$P_f^+ = \int_{-\infty}^{\infty} \mu_{x(F)}(x) dP_\Omega(x) = E_\Omega(\mu_F(x))$$

General algorithm

$$\hat{\mathbf{H}}_\Gamma : \Gamma \ni \gamma \to \hat{\mathbf{H}}_\Gamma(\gamma) \in I(R),$$

$$\mu_F(\mathbf{h}) = P_\Gamma\{\gamma : \mathbf{h} \in \hat{\mathbf{H}}(\gamma)\}.$$

$$\mathbf{L}(\mathbf{u},\mathbf{h}) = \mathbf{f}(\mathbf{h}),\ \ \mathbf{u} \in V$$

$$\mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \quad y = g(\mathbf{h})$$

$$\mu_{g(F)}(y) = \sup_{\mathbf{h}: y = g(\mathbf{h})} \mu_F(\mathbf{h})$$

$$P_f^+ = \sup_{\mathbf{h}: g(\mathbf{h})<0} \mu_F(\mathbf{h}) = \sup_{y: y<0} \mu_{g(F)}(y)$$

Other methods of modeling of uncertainty:

- TBM model (Philip Smith).
- imprecise probability (Imprecise Probability Project, Buckley, Thomas etc.).
- etc.

Numerical methods of solution of partial differential equations

Numerical methods of solution of partial differential equations

- finite element method (FEM)
- boundary element method (BEM)
- finite difference method (FDM)

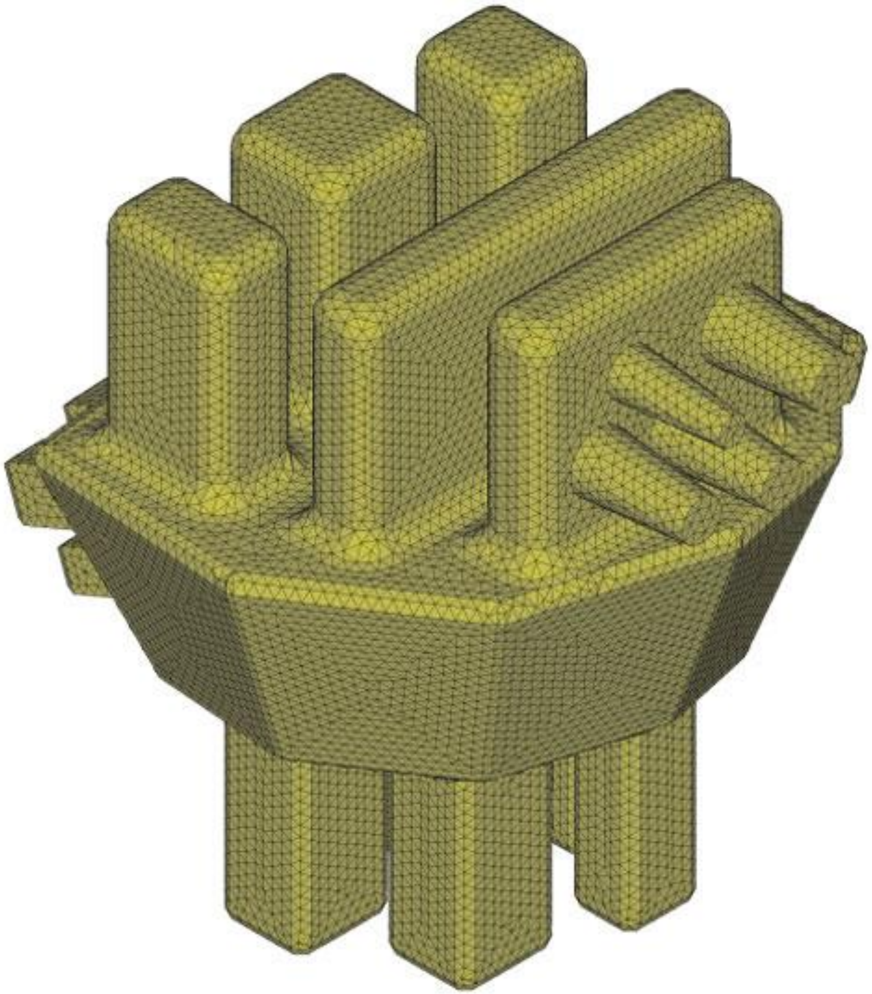
1) Boundary value problem.

2) Discretization.

3) System of algebraic equations.

4) Approximate solution.

Finite element method



Curtusy to ADINA R & D, Inc.

Using FEM
we can solve very
complicated problems.

These problems
have thousands
degree of freedom.

Algorithm

$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)= f, \quad x\in\Omega$$

$$u=0, \quad x\in\partial\Omega$$

$$-\int_\Omega \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) v\, d\Omega = \int_\Omega f v\, d\Omega$$

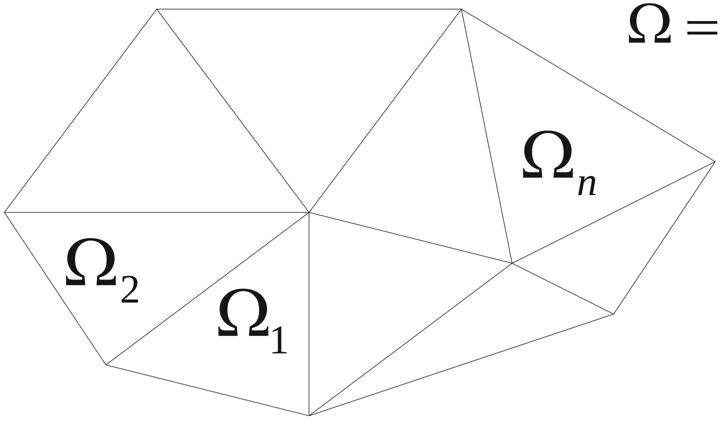
$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) d\Omega = \int_\Omega f v\, d\Omega$$

$$a(u,v) = \int_{\Omega} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \right) d\Omega$$

$$l(v) = \int_{\Omega} f v \, d\Omega$$

$$\forall v \in V,\ a(u,v) = l(v)$$

$$V_h \approx V$$



$$\Omega = \bigcup_i \Omega_i$$

$$u_h(x) = \sum_i u_i \cdot \varphi_i(x), \quad v_h(x) = \sum_i v_i \cdot \varphi_i(x)$$

$\varphi_i(x_j) = \delta_{\mathrm{ij}}$ - shape functions

$$\forall v_h \in V_h,\ a(u_h, v_h) = l(v_h)$$

System of linear algebraic equations

$$\mathbf{Ku} = \mathbf{Q}$$

$$K_{ij} = a(\varphi_i, \varphi_j), \qquad Q_i = l(\varphi_i)$$

Approximate solution

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{Q}$$

$$u_h(x) = \sum_i u_i \cdot \varphi_i(x),$$

$$u_h(x) \approx u(x)$$

Numerical methods of solution of fuzzy partial differential equations

Application of finite element method to solution of fuzzy partial differential equations.

Parameter dependent boundary value problem.

$$\mathbf{L}(\mathbf{x},\mathbf{u},\mathbf{h})=\mathbf{f}(\mathbf{x},\mathbf{h}),\quad \mathbf{u}\in V,\quad \mathbf{h}\in F$$

$$\mathbf{K}(\mathbf{h})\mathbf{u}=\mathbf{Q}(\mathbf{h}),\quad \mathbf{h}\in F$$

$$\mathbf{u}=\mathbf{u}(\mathbf{h}),\ \mathbf{h}\in F$$

$$\mathbf{u}_F\in F(R^n)$$

α-level cut method

$$\hat{\mathbf{h}}_\alpha = \{\mathbf{h} : \mu_F(\mathbf{h}) \geq \alpha\}$$

$$\boxed{\hat{\mathbf{u}}_\alpha = \{\mathbf{u} : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}_\alpha\}}$$

$$\mu_{\mathbf{u}(F)}(\mathbf{u}) = sup\{\alpha : \mathbf{u} \in \hat{\mathbf{u}}_\alpha\}$$

The same algorithm can be apply
with BEM or FDM.

$$\hat{\mathbf{u}}_{\alpha} = \{\mathbf{u} : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}_{\alpha}\}$$

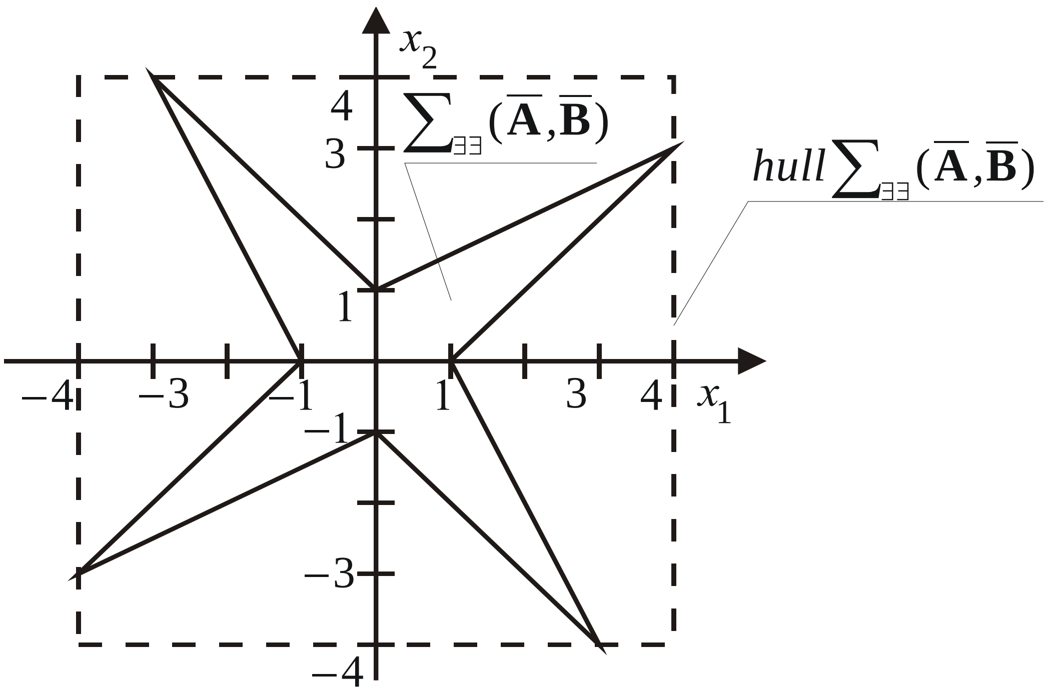
Computing accurate solution is NP-Hard.

Kreinovich V., Lakeyev A., Rohn J., Kahl P., 1998,
Computational Complexity Feasibility of Data Processing and Interval Computations.
Kluwer Academic Publishers, Dordrecht

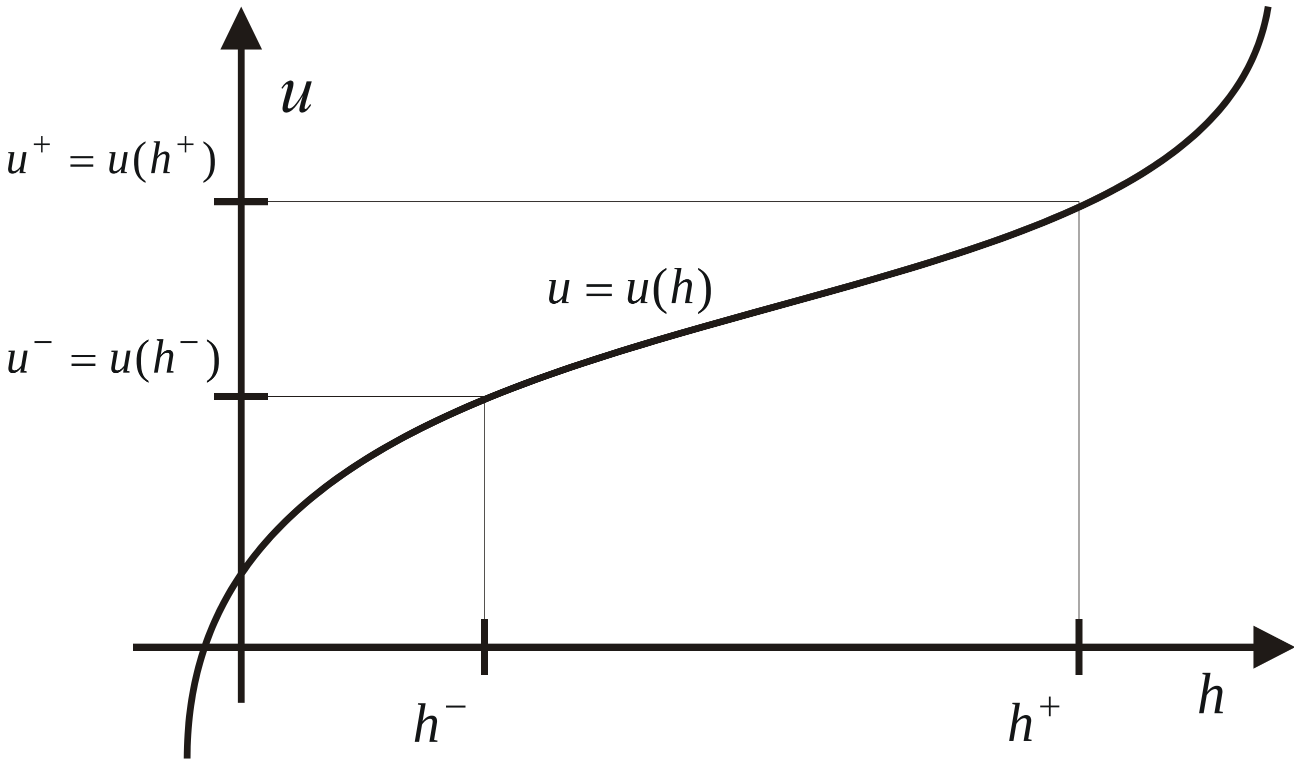
We can solve these equation only in special cases.

Solution set of system of linear interval equations is very complicated.

$$\begin{bmatrix} [1,2] & [2,4] \\ [2,4] & [1,2] \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} [-1,1] \\ [1,2] \end{bmatrix}$$



Monotone functions



$$u^- = u(h^-), \qquad u^+ = u(h^+).$$

$$\hat{\mathbf{u}}_\alpha = \{\mathbf{u} : \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}_\alpha\}$$

$$\hat{\mathbf{h}}_\alpha \subset R^m$$

2^m system equations have to be solved.

Sensitivity analysis

If $\dfrac{\partial u}{\partial h} > 0$, then $u^- = u(h^-), \quad u^+ = u(h^+)$

If $\dfrac{\partial u}{\partial h} < 0$, then $u^- = u(h^+), \quad u^+ = u(h^-)$

1+2n system of equation
(in the worst case)
have to be solved.

Multidimensional algorithm

$$\mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}_\alpha$$

$$\mathbf{K}(\mathbf{h}_0^\alpha)\frac{\partial \mathbf{u}(\mathbf{h}_0^\alpha)}{\partial h_i} = \frac{\partial \mathbf{Q}(\mathbf{h}_0^\alpha)}{\partial h_i} - \frac{\partial \mathbf{K}(\mathbf{h}_0^\alpha)}{\partial h_i}\mathbf{u}(\mathbf{h}_0^\alpha), \quad i = 1,\ldots,m$$

$$\mathbf{h}_0^\alpha = mid(\hat{\mathbf{h}}_\alpha)$$

$$\mathbf{S}_i^\alpha = \left[sign\left(\frac{\partial u_i}{\partial h_1}\right) \quad ,\ldots, \quad sign\left(\frac{\partial u_i}{\partial h_m}\right) \right], \quad i = 1,\ldots,n$$

Calculate unique sign vectors

$$\mathbf{S}_q^{\alpha*}, \quad q = 1,...,k.$$

If $\mathbf{S}_i^{\alpha} = (-1)\cdot\mathbf{S}_j^{\alpha}$, then $\mathbf{S}_i^{\alpha} \Leftrightarrow \mathbf{S}_j^{\alpha}$.

Calculate unique interval solutions

$$\hat{\mathbf{u}}_i^{\alpha*} = [\mathbf{u}(\hat{\mathbf{h}}_\alpha, \mathbf{S}_i^{\alpha*}), \mathbf{u}(\hat{\mathbf{h}}_\alpha, (-1)\cdot\mathbf{S}_i^{\alpha*})]$$

Calculate all interval solutions

$$\forall i \in \{1,...,n\}, \exists j \in \{1,...,k\}, \quad \hat{\mathbf{u}}_i^{\alpha} = \hat{\mathbf{u}}_j^{\alpha*}$$

Computational complexity

1+2*n* system of equation (in the worst case) have to be solved.

$\mathbf{s}_i^{\alpha}$

All sign vectors

Unique sign vectors

$$\frac{\partial \mathbf{u}}{\partial \mathbf{h}} = \begin{bmatrix} \frac{\partial u_1}{\partial h_1} & \cdots & \frac{\partial u_1}{\partial h_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial u_i}{\partial h_1} & \cdots & \frac{\partial u_i}{\partial h_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial h_1} & \cdots & \frac{\partial u_n}{\partial h_m} \end{bmatrix} \Rightarrow \quad \mathbf{S}^{\alpha} = \begin{bmatrix} \mathbf{S}_1^{\alpha} \\ \cdots \\ \mathbf{S}_i^{\alpha} \\ \cdots \\ \mathbf{S}_n^{\alpha} \end{bmatrix} \Rightarrow \quad \mathbf{S}^{\alpha*} = \begin{bmatrix} \mathbf{S}_1^{\alpha*} \\ \cdots \\ \mathbf{S}_q^{\alpha*} \\ \cdots \\ \mathbf{S}_k^{\alpha*} \end{bmatrix}$$

This method can be applied
only when
the relation between the solution
and uncertain parameters

$$\mathbf{u} = \mathbf{u}(\mathbf{h})$$

is monotone.

According to my experience
(and many numerical results
which was published)
in problems of computational mechanics
the intervals $\hat{\mathbf{h}}_{\alpha}$ are usually narrow
and the relation **u**=**u**(**h**)
is **<u>monotone</u>**.

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Particular case - system of linear interval equations

$$\begin{bmatrix} K_{11}^F & \dots & K_{1n}^F \\ \dots & \dots & \dots \\ K_{n1}^F & \dots & K_{nn}^F \end{bmatrix} \begin{bmatrix} X_1 \\ \dots \\ X_n \end{bmatrix} = \begin{bmatrix} Q_1^F \\ \dots \\ Q_n^F \end{bmatrix}$$

$$\begin{bmatrix} \hat{K}_{11}^\alpha & \dots & \hat{K}_{1n}^\alpha \\ \dots & \dots & \dots \\ \hat{K}_{n1}^\alpha & \dots & \hat{K}_{nn}^\alpha \end{bmatrix} \begin{bmatrix} X_1 \\ \dots \\ X_n \end{bmatrix} = \begin{bmatrix} \hat{Q}_1^\alpha \\ \dots \\ \hat{Q}_n^\alpha \end{bmatrix}$$

$$\mathbf{K}_0^\alpha \mathbf{u}_0^\alpha = \mathbf{Q}_0^\alpha$$

$$\mathbf{K}_0^\alpha \frac{\partial \mathbf{u}_0^\alpha}{\partial h_i} = \frac{\partial \mathbf{Q}_0^\alpha}{\partial h_i} - \frac{\partial \mathbf{K}_0^\alpha}{\partial h_i} \mathbf{u}_0^\alpha$$

$$\mathbf{S}_\alpha^i = \left[sign\left(\frac{\partial u_i}{\partial h_1} \right), ..., sign\left(\frac{\partial u_i}{\partial h_m} \right) \right]$$

$$\mathbf{S}_\alpha^i = \mathbf{S}_\alpha^{j*}, \quad \text{where } \; j = C_i$$

$$\hat{\mathbf{X}}_\alpha^{i*} = [\mathbf{X}(\hat{\mathbf{h}}_\alpha, \mathbf{S}_\alpha^{i*}), \mathbf{X}(\hat{\mathbf{h}}_\alpha, (-1) \cdot \mathbf{S}_\alpha^{i*})]$$

$$\hat{X}_{\alpha i} = \hat{X}_{\alpha i}^{j*}, \quad \text{where } \; j = C_i$$

Computational complexity of this algorithm

$1+2p$ - system of equations.

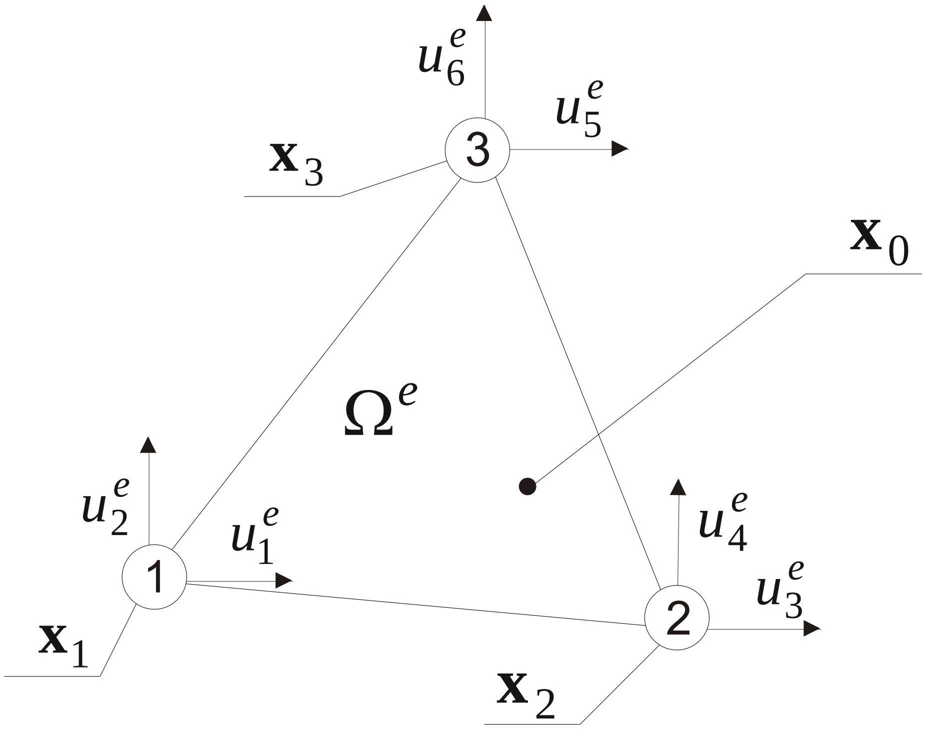
p - number of independent sign vectors $\mathbf{S}_{\alpha}^{i*}$.

$$p \in [1, n]$$

$[1+2, 1+2n]$ - system of equations

n - number of degree of freedom.

Calculation of the solution between the nodal points



$$\mathbf{u}^e(\mathbf{x}) = \mathbf{N}^e(\mathbf{x})\mathbf{u}^e$$

$$\mathbf{u}^e(\mathbf{x},\mathbf{h}) = \mathbf{N}^e(\mathbf{x},\mathbf{h})\mathbf{u}^e(\mathbf{h})$$

Extreme solution inside the element
cannot be calculated using only the nodal solutions **u**.
(because of the unknown dependency of the parameters)

Extreme solution can be calculated
using sensitivity analysis

$$\mathbf{S}^e = \left[sign\left(\frac{\partial u^e(\mathbf{x}_0,\mathbf{h}_0)}{\partial h_1} \right), \; ... \; , \; sign\left(\frac{\partial u^e(\mathbf{x}_0,\mathbf{h}_0)}{\partial h_m} \right) \right]$$

Calculation of extreme solutions between the nodal points.

1) Calculate sensitivity of the solution.
(this procedure use existing results of the calculations)

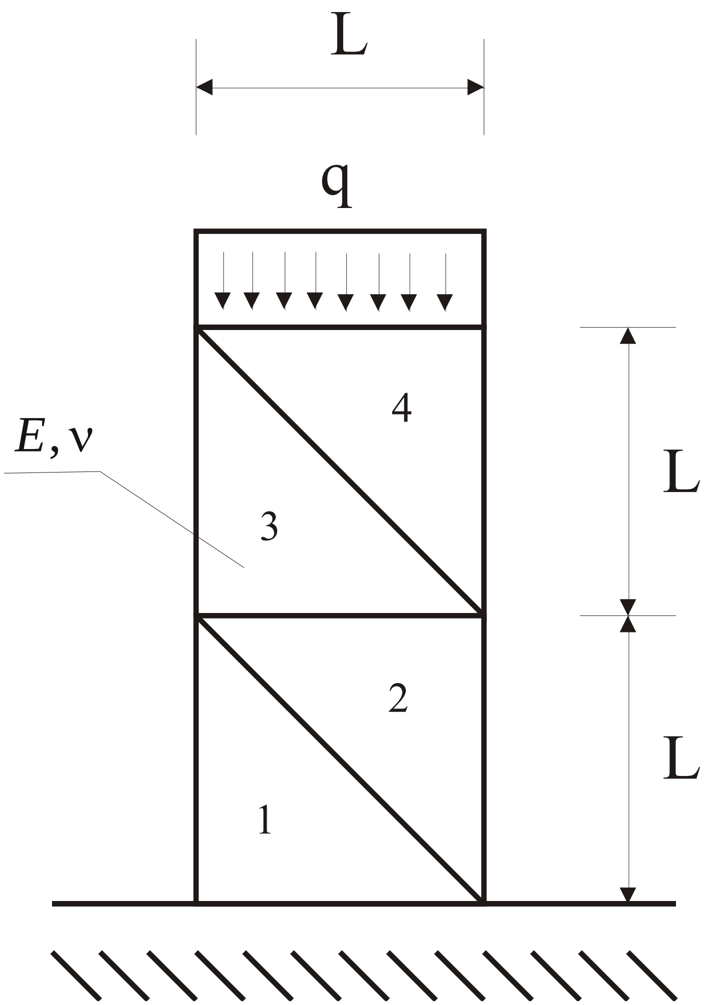
$$\mathbf{S}^e = \left[sign\left(\frac{\partial u^e(\mathbf{x}_0, \mathbf{h}_0)}{\partial h_1} \right), \ \ldots \ , \ sign\left(\frac{\partial u^e(\mathbf{x}_0, \mathbf{h}_0)}{\partial h_m} \right) \right]$$

2) If this sensitivity vector is new then calculate the new interval solution.
The extreme solution can be calculated using this solution.

3) If sensitivity vector isn't new then calculate the extreme solution using existing data.

Numerical example

Plane stress problem in theory of elasticity



Plane stress problem in theory of elasticity

$$\frac{E}{2(1+\nu)} u_{\alpha,\beta\beta} + \frac{E}{2(1-\nu)} u_{\beta,\beta\alpha} + \rho f_\alpha = 0, \quad \alpha,\beta = 1,2$$

$$u_\alpha = \overset{*}{u}_\alpha, \quad x \in \partial\Omega_u$$

$$\sigma_{\alpha\beta} n_\beta = \overset{*}{t}_\alpha, \quad x \in \partial\Omega_\sigma$$

ρ - mass density,
E,ν - material constant,
f_α - mass force.

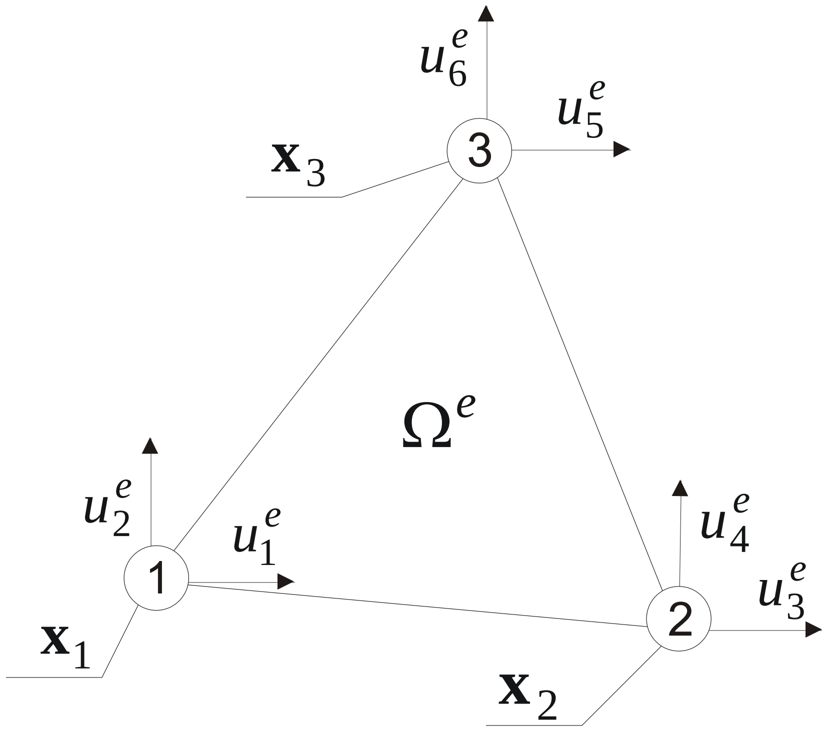
$$u_{\alpha,\beta} = \frac{\partial u_\alpha}{\partial x_\beta}$$

Finite element method

Ku=Q

$$\mathbf{K} = \int_{\Omega^e} \mathbf{B}^T \mathbf{D}\mathbf{B}\, d\Omega, \qquad \mathbf{Q} = \int_{\Omega} \mathbf{N}^T \rho \mathbf{f}\, d\Omega + \int_{\partial\Omega_\sigma} \mathbf{N}^T \mathbf{t}\, dS,$$

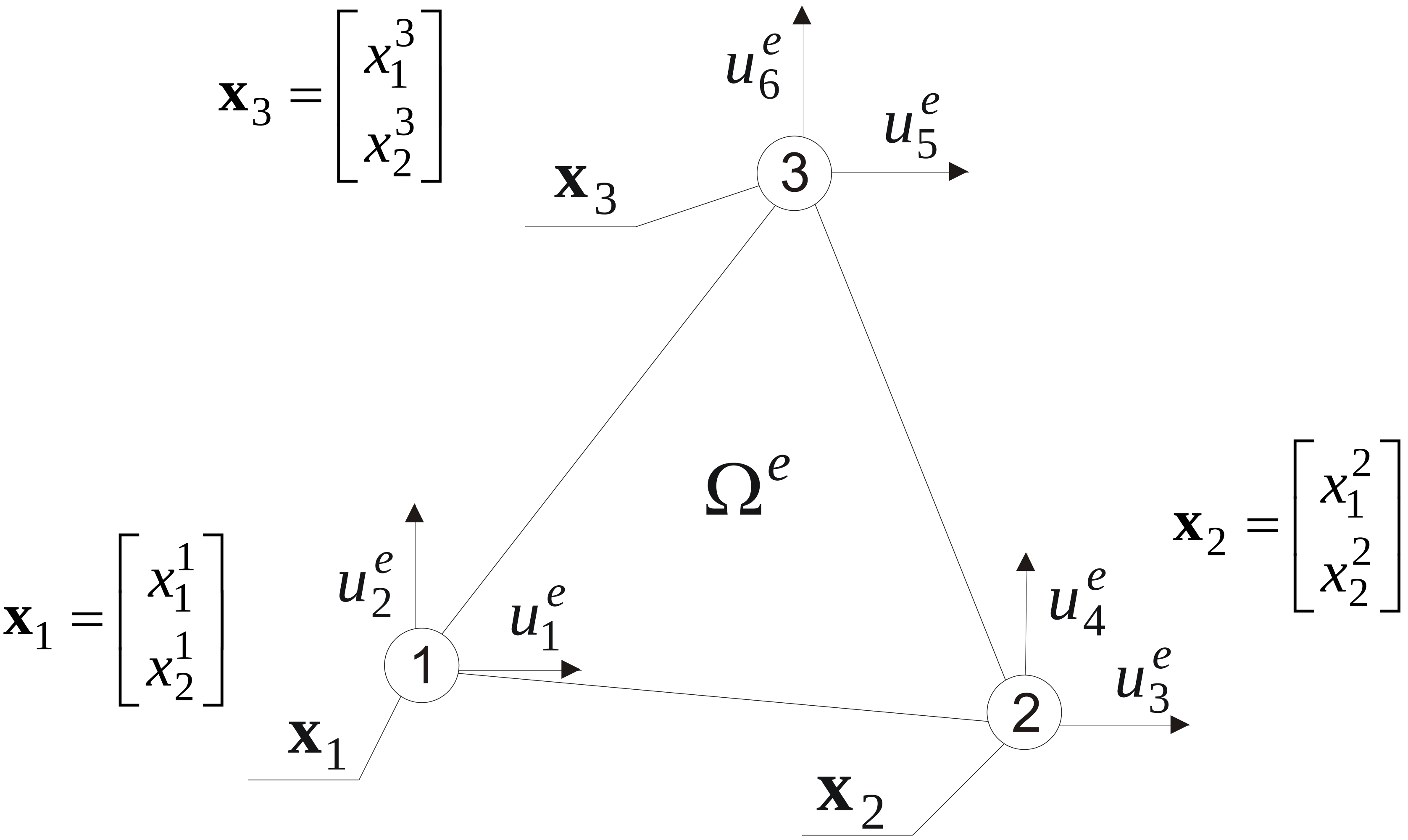
$$\mathbf{u}(x) = \mathbf{N}(x)\mathbf{u}, \qquad u_i(x) = N_{ij}(x) u_j.$$



$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{u}(x) = \mathbf{N}(\mathbf{x})\mathbf{u}$$

$$\mathbf{u}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} N_1(\mathbf{x}) & 0 & N_2(\mathbf{x}) & 0 & N_3(\mathbf{x}) & 0 \\ 0 & N_1(\mathbf{x}) & 0 & N_2(\mathbf{x}) & 0 & N_3(\mathbf{x}) \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$



$\mathbf{x}_3 = \begin{bmatrix} x_1^3 \\ x_2^3 \end{bmatrix}$
u_6^e
u_5^e
3
$\mathbf{x}_3$
Ω^e
$\mathbf{x}_2 = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$
$\mathbf{x}_1 = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}$
u_2^e
u_1^e
u_4^e
u_3^e
1
2
$\mathbf{x}_1$
$\mathbf{x}_2$

$$N_i(\mathbf{x}) = a_i + b_i x_1 + c_i x_2$$

$$N_i(\mathbf{x}_j) = \delta_{ij}$$

$$N_1(\mathbf{x}) = \frac{x_1^2 x_2^3 - x_1^3 x_2^2 + (x_2^2 - x_2^3) x_1 + (x_1^3 - x_1^2) x_2}{\Delta}$$

$$\Delta = \begin{vmatrix} 1 & x_1^1 & x_2^1 \\ 1 & x_1^2 & x_2^2 \\ 1 & x_1^3 & x_2^3 \end{vmatrix}$$

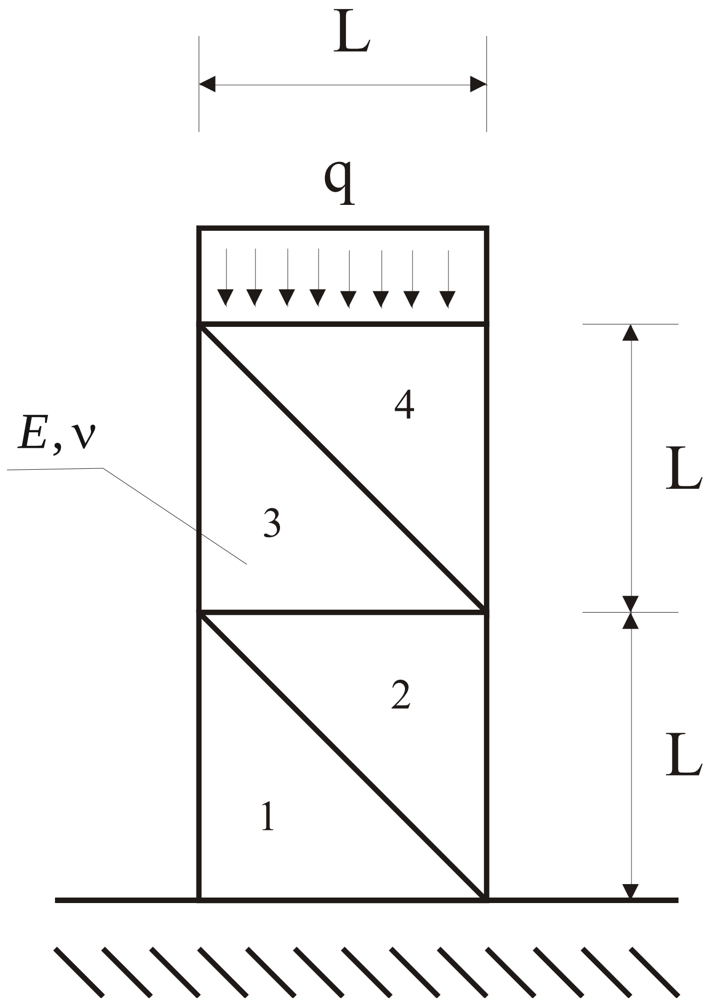
Etc.

$$\mathbf{K} = \int_{\Omega^e} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega,$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & 0 & \frac{\partial N_2}{\partial x_1} & 0 & \frac{\partial N_3}{\partial x_1} & 0 \\ 0 & \frac{\partial N_1}{\partial x_2} & 0 & \frac{\partial N_2}{\partial x_2} & 0 & \frac{\partial N_3}{\partial x_2} \\ \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_2}{\partial x_1} & \frac{\partial N_3}{\partial x_2} & \frac{\partial N_3}{\partial x_1} \end{bmatrix}$$

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Geometry of the problem



Fuzzy parameters:

$$E_1, E_2, E_3, E_4$$

Real parameters:

$$q, \nu, L$$

Numerical data

L=1 [m], $q = 1\left[\frac{kN}{m}\right]$, $\nu = 0.3.$

α	α=0	α=1
$\hat{E}_\alpha^1$	[189, 231] [GPa]	210 [GPa]
$\hat{E}_\alpha^2$	[189, 231] [GPa]	210 [GPa]
$\hat{E}_\alpha^3$	[189, 231] [GPa]	210 [GPa]
$\hat{E}_\alpha^4$	[189, 231] [GPa]	210 [GPa]

Numerical results

Fuzzy stress

α	α=0	α=1
$\hat{\sigma}_{y\alpha}^{1}$	[0.96749, 0.974493] [kPa]	0.971063 [kPa]
$\hat{\sigma}_{y\alpha}^{2}$	[1.02833, 1.02955] [kPa]	1.02894 [kPa]
$\hat{\sigma}_{y\alpha}^{3}$	[0.98086, 1.01719] [kPa]	0.999086 [kPa]
$\hat{\sigma}_{y\alpha}^{4}$	[0.982807, 1.01914] [kPa]	1.00091 [kPa]

Fuzzy displacement

Nr	$\hat{u}_{i\alpha}, \alpha = 0 \quad [m]$	Nr	$\hat{u}_{i\alpha}, \alpha = 0 \quad [m]$	Nr	$\hat{u}_{i\alpha}, \alpha = 0 \quad [m]$
1	[0, 0]	5	[3.2517e-14,7.49058e-13]	9	[-1.5134e-12,1.0498e-12]
2	[0, 0]	6	[3.81132e-12, 4.692e-12]	10	[8.1381e-12,9.9465e-12]
3	[0, 0]	7	[-1.5243e-12,-4.9879e-13]	11	[-3.1758e-12,-1.7949e-13]
4	[0, 0]	8	[4.4199e-12, 5.4275e-12]	12	[8.7620e-12,1.0709e-11]

Numerical example

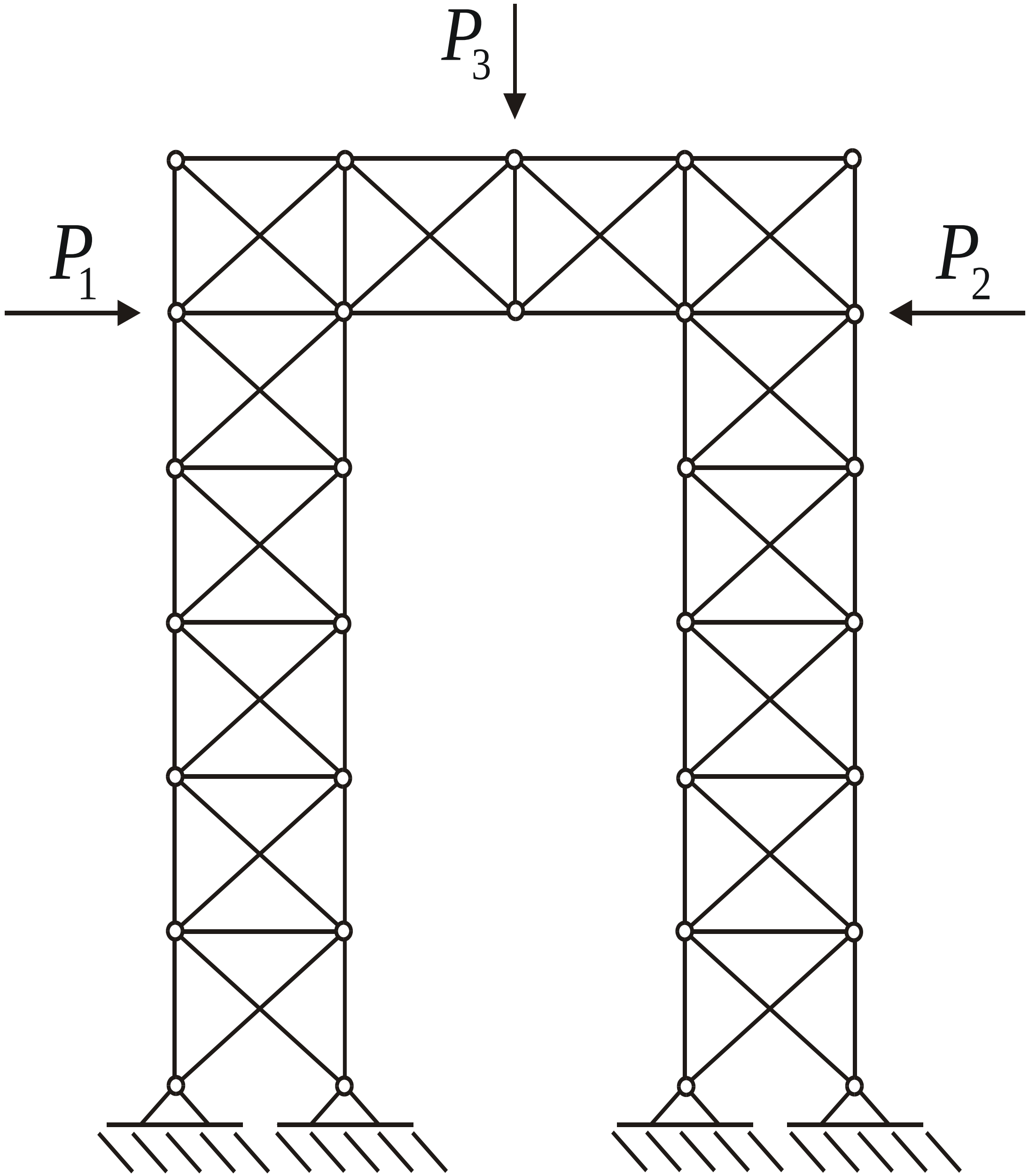
Truss structure

Numerical example (truss structure)

$$\begin{cases} \dfrac{d}{dx}\left(EA\dfrac{du}{dx} \right) + n = 0 \\ \text{Boundary conditions} \end{cases}$$

$$a(u,v) = \int_0^L EA \frac{du}{dx}\frac{dv}{dx}dx, \qquad l(v) = \int_0^L nvdx + ...,$$

$$\forall v \in V,\ a(u,v) = l(v)$$



P=10 [kN]

Young's modules
the same like
in previous example.

$\nu = 0.3$

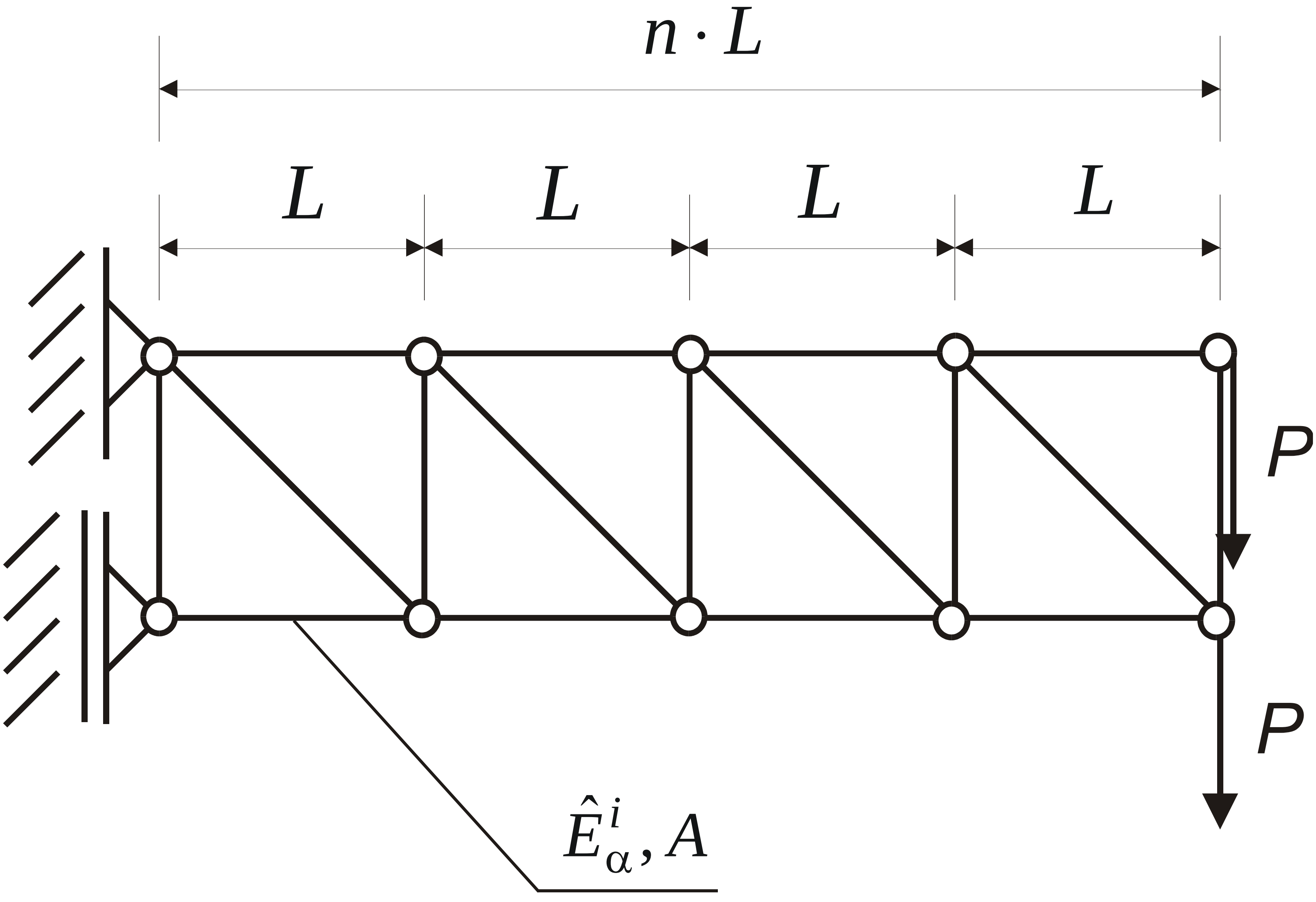
L=1 [m]

Interval solution: axial force [N]

1	[3145.34, 4393.45]	21	[-751.05, -742.133]	41	[-194.644, -208.406]	61	[1686.62, 1641.68]
2	[1482.48, 1914.16]	22	[453.902, 470.55]	42	[-2188.83, -2205.43]	62	[1528.04, 1545.77]
3	[-172.138, -221.845]	23	[-1417.47, -1433.55]	43	[275.268, 294.73]	63	[-343.334, -358.339]
4	[164.454, 279.737]	24	[6437.89, 6417.04]	44	[-7448.38, -7428.59]	64	[2470.18, 2524.72]
5	[-958.619, -936.417]	25	[-7444.75, -7432.58]	45	[-194.644, -208.406]	65	[-947.416, -949.597]
6	[2459.35, 2536.53]	26	[-200.408, -202.065]	46	[6417.52, 6439.45]	66	[253.654, 185.319]
7	[1527.83, 1546.14]	27	[-2196.2, -2197.33]	47	[451.658, 473.02]	67	[1683.18, 1701.27]
8	[-343.544, -357.966]	28	[283.42, 285.763]	48	[-1419.72, -1431.08]	68	[-188.192, -202.832]
9	[1708.72, 1617.27]	29	[4020.01, 4013.59]	49	[-738.486, -755.954]	69	[3683.74, 3761.16]
10	[-840.883, -841.035]	30	[-200.408, -202.065]	50	[-166.773, -171.028]		
11	[1132.62, 1189.25]	31	[-9461.8, -9431.91]	51	[4242.96, 4244.56]		
12	[1532.73, 1547.37]	32	[3589.87, 3583.79]	52	[1655.57, 1672.95]		
13	[-338.641, -356.736]	33	[-3488.96, -3478.74]	53	[-215.805, -231.149]		
14	[3028.51, 2962.81]	34	[713.715, 704.035]	54	[-266.518, -258.031]		
15	[-932.071, -929.76]	35	[4929.89, 4924.37]	55	[-930.146, -931.887]		
16	[-278.358, -245.009]	36	[720.439, 696.638]	56	[3007.62, 2985.78]		
17	[1656.79, 1671.62]	37	[3580.36, 3594.25]	57	[1531.23, 1549.04]		
18	[-214.586, -232.489]	38	[-3482.95, -3485.36]	58	[-340.144, -355.068]		
19	[4264.06, 4221.36]	39	[-9466.06, -9427.23]	59	[1144.66, 1176]		
20	[-169.222, -168.335]	40	[4010.55, 4024]	60	[-839.969, -841.95]		

Truss structure

(Second example)



$n \cdot L$
L
L
L
L
P
P
$\hat{E}_{\alpha}^{i}, A$

Data

$$\hat{E}_\alpha = [189, 231][GPa], \quad \alpha = 0,$$

$$\hat{E}_\alpha = [210, 210][GPa], \quad \alpha = 1,$$

$$L = 1[m],$$

$$A = 0.0001\,[m^2],$$

$$\hat{P}_\alpha = [9, 11]\ [\text{kN}], \quad \alpha = 0,$$

$$\hat{P}_\alpha = [10, 10]\ [\text{kN}], \quad \alpha = 1.$$

Time of calculation

n	DOF	Elements	Time
200	804	1000	00:02:38
300	1204	1500	00:08:56
400	1604	2000	00:20:46
500	2004	2500	00:39:45

Processor: AMD Duron 750 MHz

RAM: 256 MB

Monotonicity tests

(point tests)

Monotone solutions.
(Special case)

$$\mathbf{Ku} = \mathbf{Q}(\mathbf{h}) = \sum_j \boldsymbol{\alpha}_j h_j$$

$$Q_i(\mathbf{h}) = \sum_j \alpha_{ij} h_j, \quad \alpha_{ij} \in R$$

$$\frac{\partial \mathbf{Q}}{\partial h_j} = \begin{bmatrix} \alpha_{1j} \\ \ldots \\ \alpha_{nj} \end{bmatrix} = \boldsymbol{\alpha}_j = const$$

$$\frac{\partial \mathbf{K}}{\partial h_j} = \mathbf{0}$$

$$\frac{\partial \mathbf{u}}{\partial h_j} = \mathbf{K}^{-1}\left(\frac{\partial \mathbf{Q}}{\partial h_j} - \frac{\partial \mathbf{K}}{\partial h_j}\mathbf{q}\right) = \mathbf{K}^{-1}\boldsymbol{\alpha}_j = const$$

$\mathbf{u} = \mathbf{u}(\mathbf{h})$ - linear function.

Natural interval extension

$$f(x) = x^2 - x,$$

$$\hat{f}(\hat{x}) = \hat{x}^2 - \hat{x}$$

$$\hat{f}([-1, 2]) = [-1, 2] \cdot [-1, 2] - [-1, 2] =$$

$$= [-2,4] + [-2,1] = [-4,5]$$

$$f([-1, 2]) = \left[-\frac{1}{4}, 2 \right]$$

$$f(\hat{x}) \subseteq \hat{f}(\hat{x})$$

Monotonicity tests

$$\frac{\partial u(\mathbf{h})}{\partial h_i} = \frac{\partial u(\mathbf{h}_0)}{\partial h_i} + \sum_{j=1}^{m} \frac{\partial^2 u(\mathbf{h}_0)}{\partial h_i \partial h_j}(h_j - h_j^0)$$

If

$$0 \notin \frac{\partial \hat{u}(\hat{\mathbf{h}}_\alpha)}{\partial h_i} = \frac{\partial u(\mathbf{h}_0)}{\partial h_i} + \sum_{j=1}^{m} \frac{\partial^2 u(\mathbf{h}_0)}{\partial h_i \partial h_j}(\hat{h}_{j\alpha} - h_j^0)$$

then function

$$u = u(\mathbf{h})$$

is monotone.

High order monotonicity tests

$$\frac{\partial u(\mathbf{h})}{\partial h_i} = \frac{\partial u(\mathbf{h}_0)}{\partial h_i} + \sum_{j=1}^{m} \frac{\partial^2 u(\mathbf{h}_0)}{\partial h_i \partial h_j}(h_j - h_j^0) + \frac{1}{2}\sum_{j}^{m}\sum_{k}^{m} \frac{\partial^2 u(\mathbf{h}_0)}{\partial h_i \partial h_j}(h_j - h_j^0)(h_k - h_k^0) + \ldots$$

If

$$0 \notin \frac{\partial \hat{u}(\hat{\mathbf{h}}_\alpha)}{\partial h_i} = \frac{\partial u(\mathbf{h}_0)}{\partial h_i} + \sum_{j=1}^{m} \frac{\partial^2 u(\mathbf{h}_0)}{\partial h_i \partial h_j}(\hat{h}_{j\alpha} - h_j^0) + \ldots$$

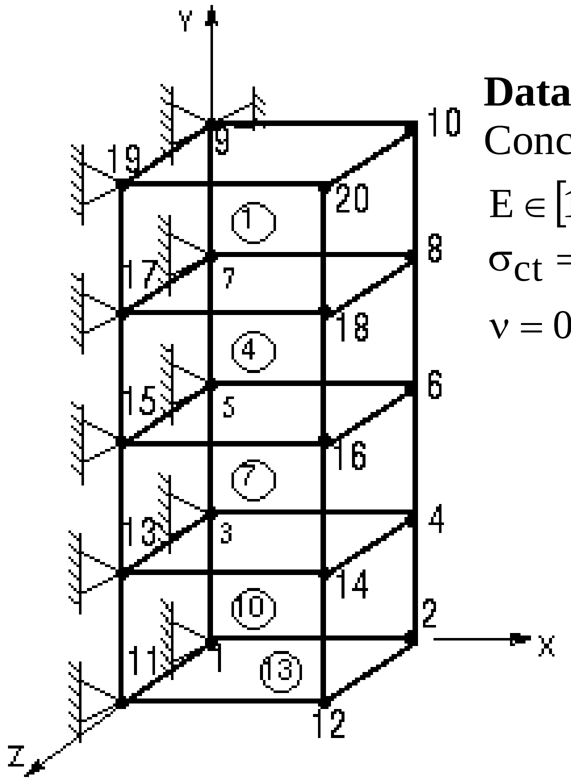
then function

$$u = u(\mathbf{h})$$

is monotone.

Numerical example

(Reinforced Concrete Beam)



Representative Finite Element Model

Data

Concrete

$E \in [1.3, 1.5] \cdot 10^4$ MPa

$\sigma_{ct} = 0$ MPa

$\nu = 0$

Steel

$E \in [2.0, 2.2] \cdot 10^5$ MPa

$\nu \in [0.2, 0.3]$

Geometry

$a = 0.127$ m

$b = 0.152$ m

$A = 0.019\,m^2$

Numerical result

$$\alpha = 0: \quad u_{2x} \in [0.182, 0.200] \cdot 10^{-4} [m]$$

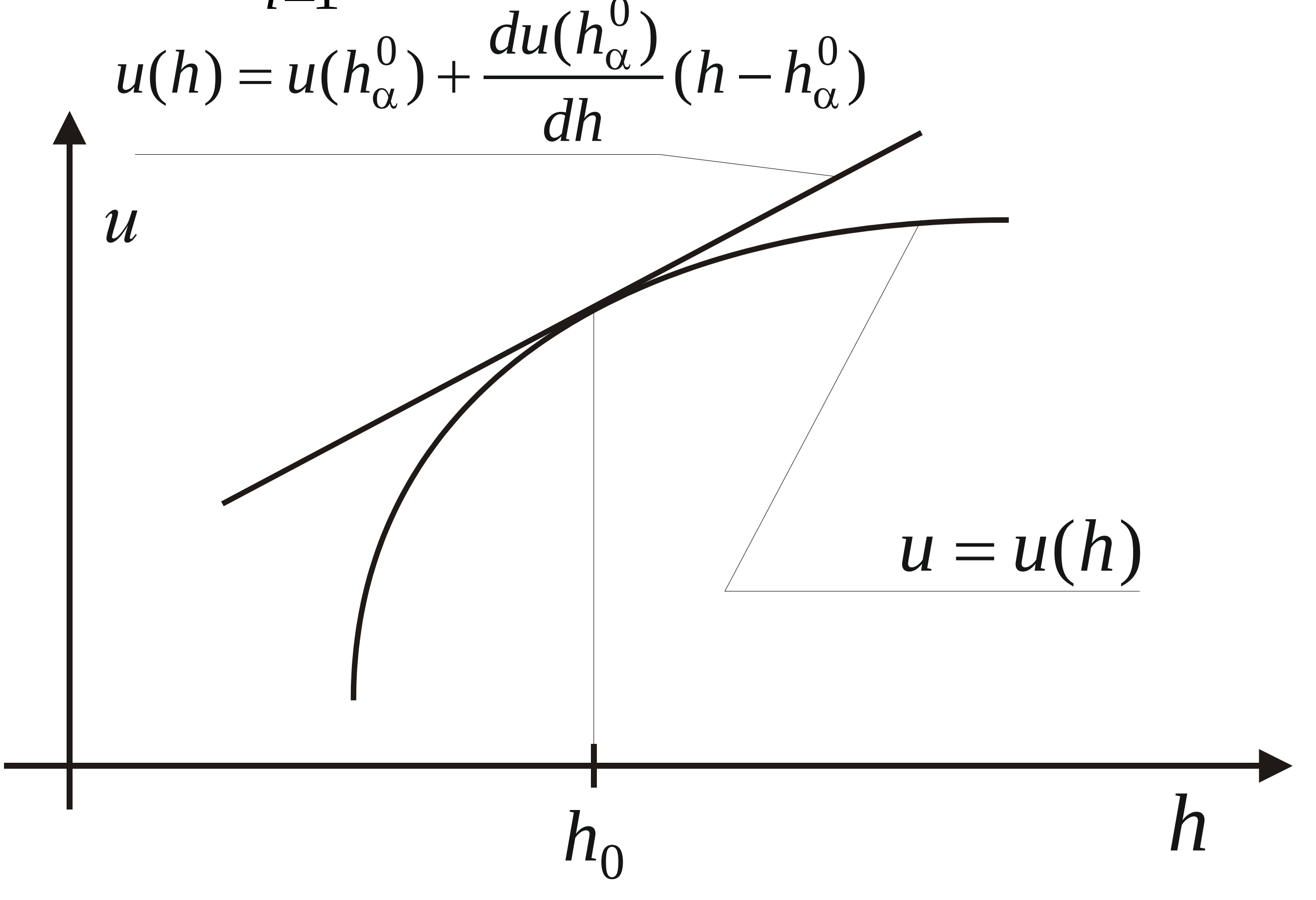
$$\alpha = 1: \quad u_{2x} \in [0.190, 0.190] \cdot 10^{-4} [m]$$

In this example
commercial FEM program ANSYS
was applied.

Point monotonicity test can be applied
to results which were generated
by the existing engineering software.

Taylor model

$$u(\mathbf{h}) = u(\mathbf{h}_\alpha^0) + \sum_{i=1}^{m} \frac{\partial u(\mathbf{h}_\alpha^0)}{\partial h_i}\left(h_i - h_{\alpha i}^0\right), \qquad \mathbf{h}_\alpha^0 = mid(\hat{\mathbf{h}}_\alpha)$$



Approximate interval solution

$$\hat{u}_\alpha = \hat{u}(\hat{\mathbf{h}}_\alpha) = u(\mathbf{h}_\alpha^0) + \sum_{i=1}^{m} \frac{\partial u(\mathbf{h}_\alpha^0)}{\partial h_i}\left(\hat{h}_{\alpha i} - h_{\alpha i}^0\right),$$

$$\hat{u}_\alpha \approx u(\hat{\mathbf{h}}_\alpha).$$

Computational complexity

$u(\mathbf{h}_\alpha^0)$ - 1 solution of $\mathbf{K}^{-1}$

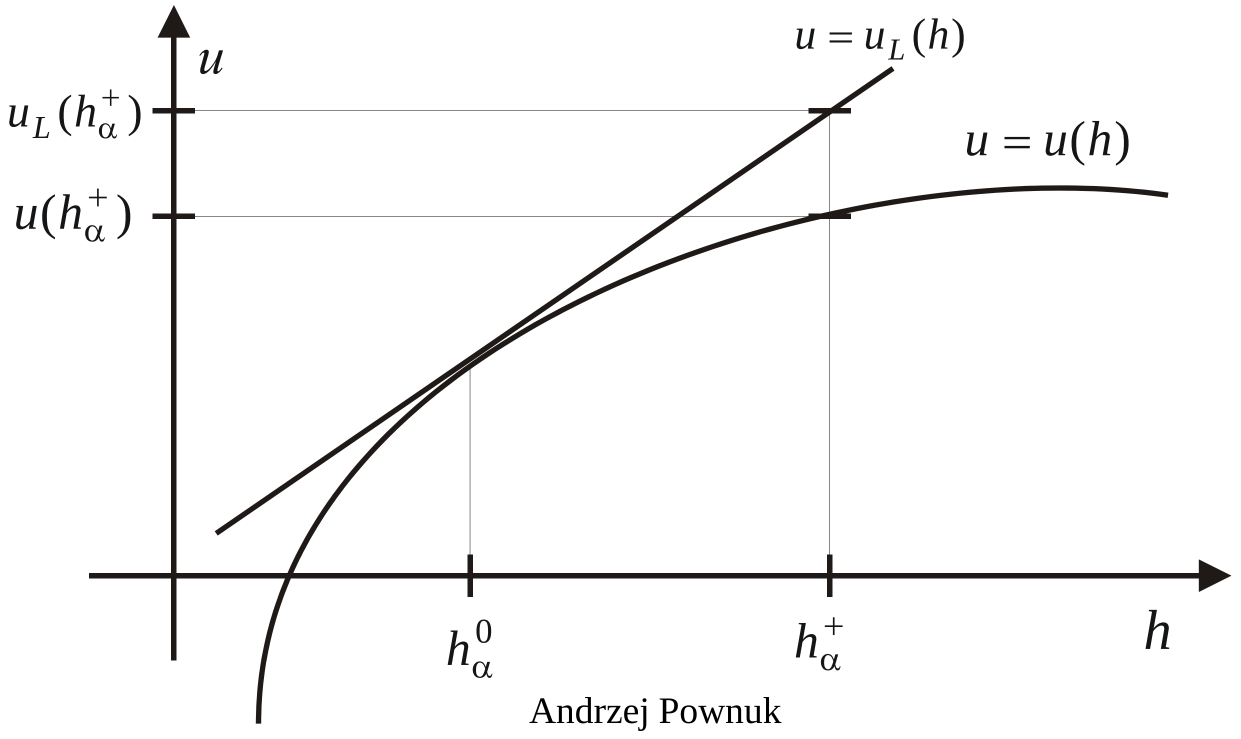
$\dfrac{\partial u(\mathbf{h}_\alpha^0)}{\partial h_i}$ - the same matrix $\mathbf{K}^{-1}$

1 - point solution

Akapan U.O., Koko T.S., Orisamolu I.R., Gallant B.K.,
Practical fuzzy finite element analysis of structures.
Finite Element in Analysis and Design, Vol. 38, 2001, pp. 93-111

$$u_L(\mathbf{h}) = u(\mathbf{h}_0) + \sum_i \frac{\partial u(\mathbf{h}_0)}{\partial h_i}(h_i - h_i^0) + \frac{1}{2}\sum_i \sum_j \frac{\partial u(\mathbf{h}_0)}{\partial h_i \partial h_j}(h_i - h_i^0)(h_j - h_j^0)$$

$$u_L(\mathbf{h}) \approx u(\mathbf{h})$$



Finite difference method

$$\frac{du(h_\alpha^0)}{dx} \approx \frac{u(h_\alpha^0 + \Delta h) - u(h_\alpha^0 - \Delta h)}{2\Delta h}$$

$$\frac{d^2u(h_\alpha^0)}{dx^2} \approx \frac{u(h_\alpha^0 + \Delta h) - 2 \cdot u(h_\alpha^0) + u(h_\alpha^0 - \Delta h)}{(\Delta h)^2}$$

$$\frac{du(h)}{dx} \approx \frac{du(h_\alpha^0)}{dx} + \frac{d^2u(h_\alpha^0)}{dx^2}(h - h_\alpha^0)$$

Monotonicity test based on finite difference method (1D)

$$\frac{du(h)}{dx} \approx \frac{du(h_\alpha^0)}{dx} + \frac{d^2u(h_\alpha^0)}{dx^2}(h - h_\alpha^0) = 0$$

$$h = h_\alpha^0 - \frac{\dfrac{du(h_\alpha^0)}{dx}}{\dfrac{d^2u(h_\alpha^0)}{dx^2}} = h_\alpha^0 - \frac{\left[u(h_\alpha^0 + \Delta h) - u(h_\alpha^0 - \Delta h)\right]\Delta h}{u(h_\alpha^0 + \Delta h) - 2 \cdot u(h_\alpha^0) + u(h_\alpha^0 - \Delta h)}$$

If $h \notin \hat{h}_\alpha$ function is monotone.

Monotonicity test based on finite differences and interval extension (1D)

$$\hat{u}_{\alpha}^{(1)} = \frac{du(\hat{h}_{\alpha})}{dx} = \frac{du(h_{\alpha}^{0})}{dx} + \frac{d^{2}u(h_{\alpha}^{0})}{dx^{2}}(\hat{h}_{\alpha} - h_{\alpha}^{0})$$

If $\hat{u}_{\alpha}^{(1)} \not\ni 0$,

then function $u = u(h)$ is monotone.

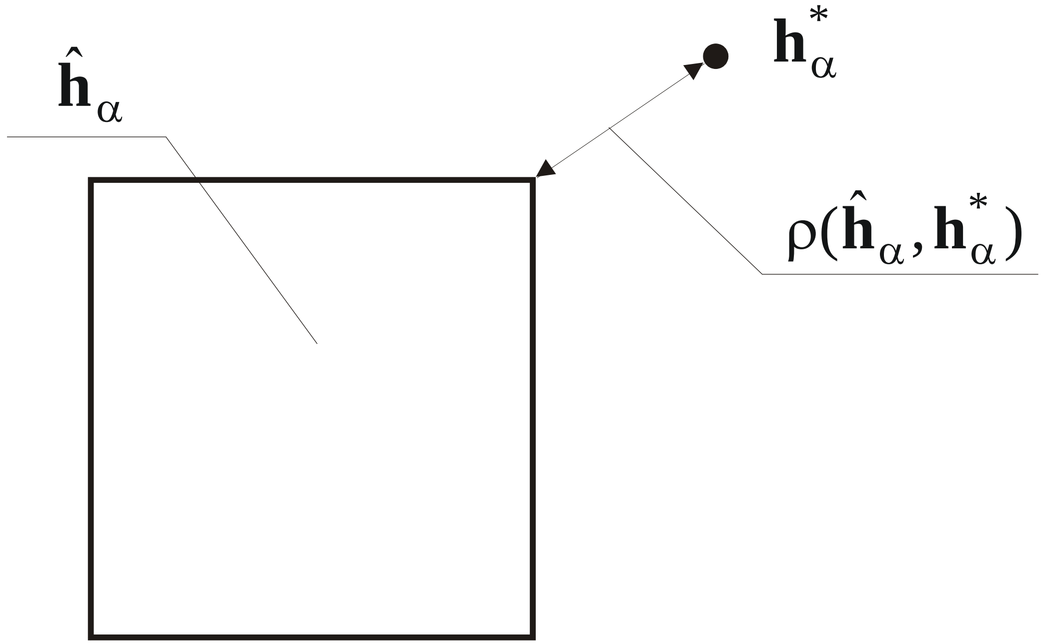
Monotonicity test based on finite difference method (multidimensional case)

$$\frac{\partial u(\mathbf{h}_\alpha^0)}{\partial h_i} + \sum_{j=1}^{m} \frac{\partial^2 u(\mathbf{h}_\alpha^0)}{\partial h_i \partial h_j} \left(h_{\alpha j}^* - h_{\alpha j}^0 \right) = 0, \quad i = 1,\ldots, m$$

$$\Downarrow$$

$$h_{\alpha i}^* \notin \hat{h}_{\alpha i}, \quad i = 1,\ldots, m$$

We can check how reliable this method is.



$$\rho(\hat{\mathbf{h}}_\alpha, \mathbf{h}^*_\alpha) >> \rho(\hat{\mathbf{h}}_\alpha) = \sup_{\mathbf{h}_1, \mathbf{h}_2 \in \hat{\mathbf{h}}_\alpha} \left\| \mathbf{h}_2 - \mathbf{h}_1 \right\|$$

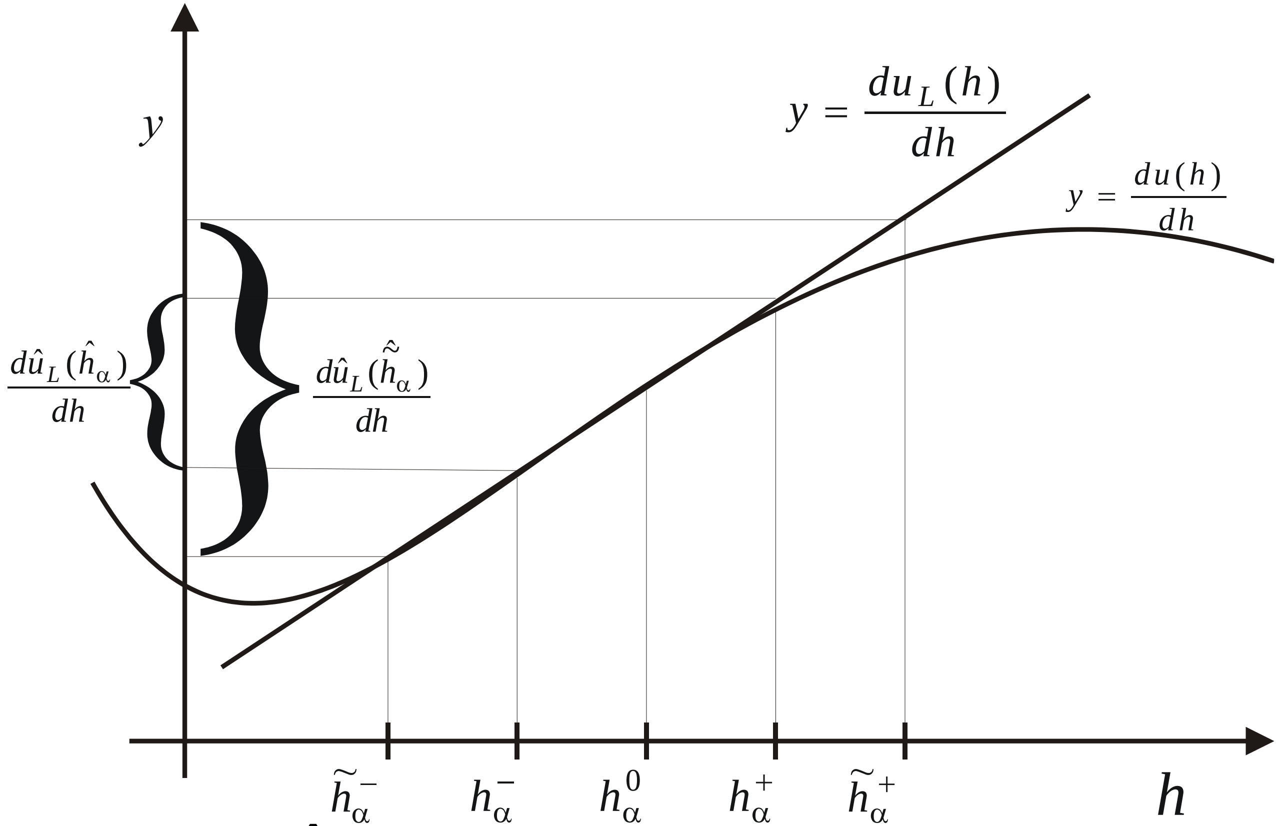
Monotonicity test based on finite differences and interval extension (multidimensional case)

$$\hat{u}_{\alpha i}^{(1)} = \frac{\partial u(\hat{\mathbf{h}}_{\alpha})}{\partial h_i} = \frac{\partial u(\mathbf{h}_{\alpha}^{0})}{\partial h_i} + \sum_{j=1}^{m} \frac{\partial^2 u(\mathbf{h}_{\alpha}^{0})}{\partial h_i \partial h_j} \left(\hat{h}_{\alpha j} - h_{\alpha j}^{0} \right)$$

$$0 \notin \hat{u}_{\alpha i}^{(1)}, \quad i = 1,\ldots,m$$

In this procedure we don't have to solve any equation.

More reliable monotonicity test



$$\hat{\tilde{h}}_\alpha \supset \hat{h}_\alpha \Rightarrow \frac{d\hat{u}_L(\hat{\tilde{h}}_\alpha)}{dh} \supset \frac{d\hat{u}_L(\hat{h}_\alpha)}{dh}$$

$$0 \notin \frac{d\hat{u}_L(\hat{\tilde{h}}_\alpha)}{dh}$$

Subdivision

$$0 \in \frac{d\hat{u}_L(\hat{h}_\alpha)}{dh}$$

$$\downarrow$$

$$\hat{h}_\alpha^1 \cup \hat{h}_\alpha^2 = \hat{h}_\alpha$$

$$0 \in \frac{d\hat{u}_L(\hat{h}_\alpha^1)}{dh} \qquad 0 \notin \frac{d\hat{u}_L(\hat{h}_\alpha^2)}{dh}$$

$$\downarrow$$

$$\hat{h}_\alpha^{11} \cup \hat{h}_\alpha^{12} = \hat{h}_\alpha^1$$

$$0 \notin \frac{d\hat{u}_L(\hat{h}_\alpha^{11})}{dh} \qquad 0 \in \frac{d\hat{u}_L(\hat{h}_\alpha^{12})}{dh} \longrightarrow \hat{h}_\alpha^{121} \cup \hat{h}_\alpha^{122} = \hat{h}_\alpha^{12}$$

If width of the interval $\hat{h}_\alpha$ i.e.

$$w(\hat{h}_\alpha) = h_\alpha^+ - h_\alpha^-$$

is sufficiently small,
then extreme values of the function u
can be approximated by using
the endpoints of given interval $\hat{h}_\alpha$.

$$u_\alpha^- = min\{u(h_\alpha^-), u(h_\alpha^+)\}, \quad u_\alpha^+ = min\{u(h_\alpha^-), u(h_\alpha^+)\}.$$

Exact monotonicity tests based on the interval arithmetic

$$\mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \quad \mathbf{h} \in \hat{\mathbf{h}}_\alpha$$

$$\mathbf{K}(\hat{\mathbf{h}}_\alpha)\mathbf{u} = \mathbf{Q}(\hat{\mathbf{h}}_\alpha)$$

$$\hat{\mathbf{K}}(\hat{\mathbf{h}}_\alpha)\frac{\partial \mathbf{u}}{\partial h_i} = \frac{\partial \hat{\mathbf{Q}}(\hat{\mathbf{h}}_\alpha)}{\partial h_i} - \frac{\partial \mathbf{K}(\hat{\mathbf{h}}_\alpha)}{\partial h_i}\hat{\mathbf{u}}(\hat{\mathbf{h}}_\alpha)$$

$$0 \notin \frac{\partial \hat{u}_i(\hat{\mathbf{h}}_\alpha)}{\partial h_j}$$

Numerical example

$$\begin{cases} R_1 < r < R_2 : & \frac{1}{r}\frac{d}{dr}\left(r\lambda \frac{dT(r)}{dr} \right) + Q = 0 \\ r = R_1 & : \ -\lambda \frac{dT(r)}{dr} = \alpha\left(T(r) - T_b\right) \\ r = R_2 & : \ T(r) = T_t \end{cases}$$

$$\frac{\partial \hat{\mathbf{T}}\left(\overline{\lambda}\right)}{\partial \lambda} = \left[\frac{\partial \hat{T}_1\left(\overline{\lambda}\right)}{\partial \lambda}, \ldots, \frac{\partial \hat{T}_n\left(\overline{\lambda}\right)}{\partial \lambda} \right]^T$$

$$\frac{\partial \hat{\mathbf{T}}\left(\overline{\lambda}\right)}{\partial \lambda} = hull\left(\sum_{\exists\exists} \left(\hat{\mathbf{K}}\left(\overline{\lambda}\right), \frac{\partial \hat{\mathbf{Q}}\left(\overline{\lambda}\right)}{\partial \lambda} - \frac{\partial \hat{\mathbf{K}}\left(\overline{\lambda}\right)}{\partial \lambda} \hat{\mathbf{T}}\left(\overline{\lambda}\right) \right) \right)$$

$$R_1 = 0.0005 \text{ [m]},\ R_2 = 10 \cdot R_1,\ \alpha = 2000 \left[\frac{W}{m^2 K}\right],\ T_b = 32 \ [{}^oC],$$

$$T_t = 37 \ [{}^oC],\ Q = 10245 \text{ [W/} m^3\text{]},\ \lambda \in [0.21,\ 0.23] \left[\frac{W}{m \cdot K}\right]$$

	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
T_i^- °C	36.586	35.470	34.782	34.284	33.894	33.573	33.302	33.065	32.857	32.669	32.500
T_i^+ °C	36.619	35.494	34.800	34.298	33.905	33.582	33.308	33.070	32.859	32.671	32.500

Sometimes system of algebraic equations is nonlinear.

$$\mathbf{K}(\mathbf{h},\mathbf{u})\mathbf{u} = \mathbf{Q}(\mathbf{h})$$

In this case we can apply interval Jacobean matrices.

$$\mathbf{F}(\mathbf{u},\mathbf{h})=\mathbf{0}$$

$$\frac{\partial \mathbf{F}(\mathbf{u},\mathbf{h})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial h_i}+\frac{\partial \mathbf{F}(\mathbf{u},\mathbf{h})}{\partial h_i}=\mathbf{0},\ \ i=1,...,m$$

$$\frac{\partial u_i}{\partial h_j}=-\frac{\left\|\begin{bmatrix}\frac{\partial F_1}{\partial u_1} & ... & \frac{\partial F_1}{\partial u_{i-1}} & \frac{\partial F_1}{\partial h_j} & \frac{\partial F_1}{\partial u_{i-1}} & ... & \frac{\partial F_1}{\partial u_n}\\ ... & ... & ... & ... & ... & ... & ...\\ \frac{\partial F_n}{\partial u_{11}} & ... & \frac{\partial F_n}{\partial u_{i-1}} & \frac{\partial F_n}{\partial h_j} & \frac{\partial F_n}{\partial u_{i+1}} & ... & \frac{\partial F_n}{\partial u_n}\end{bmatrix}\right\|}{\left|\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\right|}$$

$$\frac{\partial u_i}{\partial h_j} = -\frac{\left|\dfrac{\partial \mathbf{F}}{\partial(u_1,\ldots,u_{i-1},h_j,u_{i+1},\ldots,u_n)}\right|}{\left|\dfrac{\partial \mathbf{F}}{\partial \mathbf{u}}\right|}$$

$$sign\left(\left|\frac{\partial \mathbf{F}}{\partial \mathbf{u}}\right|\right) = const, \quad sign\left(\left|\frac{\partial \mathbf{F}}{\partial(u_1,\ldots,u_{i-1},h_j,u_{i+1},\ldots,u_n)}\right|\right) = const$$

$$\boxed{sign\left(\frac{\partial u_i}{\partial h_j}\right) = const}$$

$$\forall \mathbf{h} \in \hat{\mathbf{h}}_\alpha, \quad \left| \frac{\partial \mathbf{F}(\mathbf{x}, \mathbf{u}(\mathbf{h}), \mathbf{h})}{\partial \mathbf{u}} \right| \in \left| \frac{\partial \hat{\mathbf{F}}(\mathbf{x}, \mathbf{u}(\hat{\mathbf{h}}_\alpha), \hat{\mathbf{h}}_\alpha)}{\partial \mathbf{u}} \right|$$

Regular interval matrix

$$\forall \mathbf{A} \in \hat{\mathbf{A}}, \quad |\mathbf{A}| \neq 0.$$

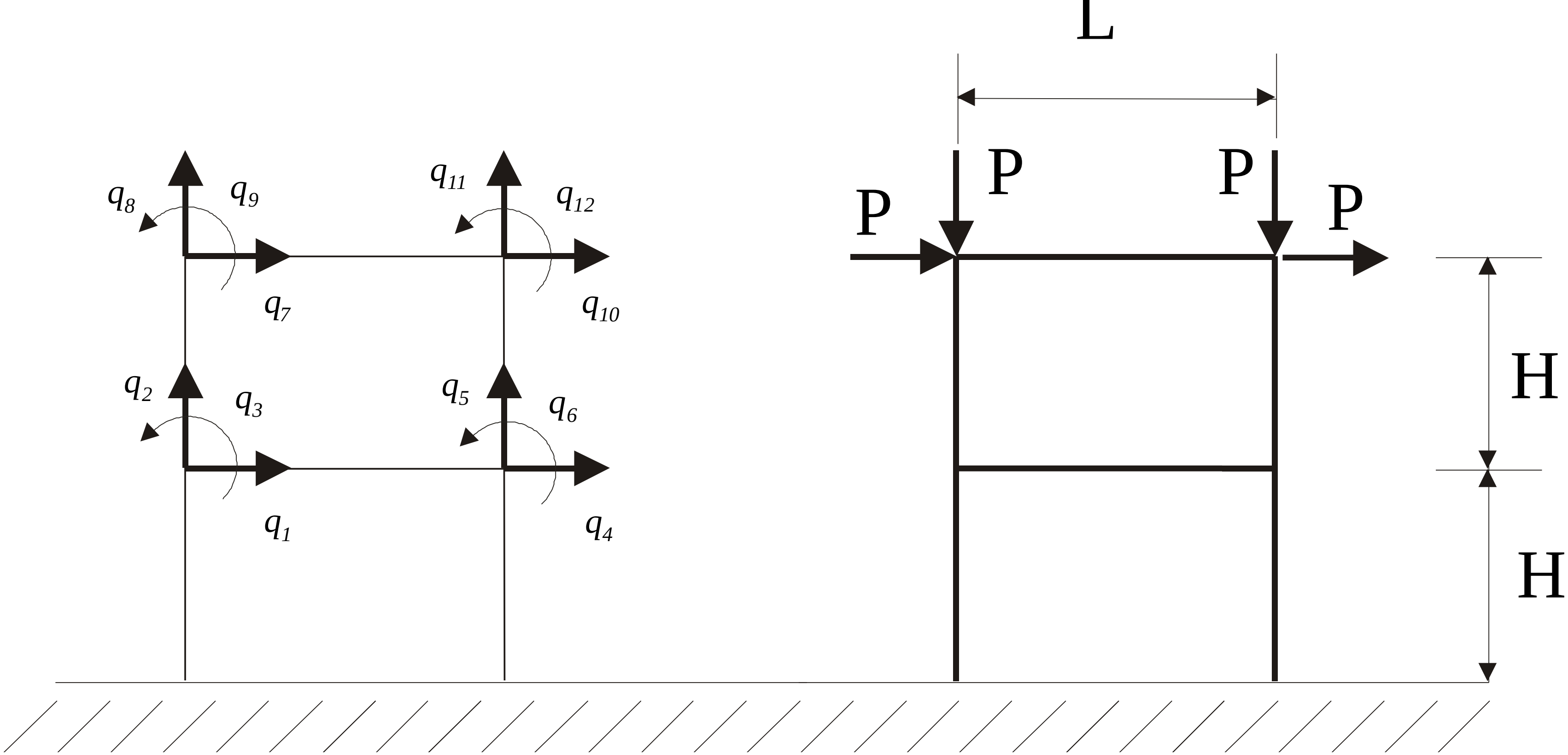
$$\mathbf{F}(\mathbf{u},\mathbf{h}) = \mathbf{0}, \quad \mathbf{h} \in \hat{\mathbf{h}}_\alpha$$

$$\frac{\partial \hat{\mathbf{F}}\left(\hat{\mathbf{u}}_\alpha, \hat{\mathbf{h}}_\alpha\right)}{\partial \mathbf{h}}$$

$$\frac{\partial \hat{\mathbf{F}}\left(\hat{\mathbf{u}}_\alpha, \hat{\mathbf{h}}_\alpha\right)}{\partial (u_1,...,u_{i-1},h_j,u_{j+1},...,u_n)}$$

It can be shown that if the following interval Jacobean matrices are regular, then solutions of parameter dependent system of equations are monotone.

Numerical example



Uncertain parameters: *E*,*A*,*J*.

Equilibrium equations of rod structures

$$\frac{d^2}{dx^2}\left(EJ\,\frac{d^2u}{dx^2} \right) = q(x)$$

$$a(u,v) = \int_0^L EJ\,\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\,dx,\;\; l(v) = \int_0^L qv\,dx + ...$$

$$\forall v \in V,\;\; a(u,v) = l(v)$$

$$E \in [210, 220]\,[GPa], \qquad J \in \left[\frac{0.05^4}{12}, \frac{0.055^4}{12}\right] [m^4],$$

$$A \in [0.05^2, 0.055^2]\,[m^2], \quad L = H = 1\ [m], \quad P = 1\ [kN].$$

	q_1 [m]	q_2 [m]	q_3	q_4 [m]	q_5 [m]	q_6
q_i^-	0.035716	0.000008	-0.011230	0.035716	-0.000021	-0.011230
q_i^+	0.037414	0.000009	-0.010718	0.037414	-0.000017	-0.010718
	q_7 [m]	q_8 [m]	q_9	q_{10} [m]	q_{11} [m]	q_{12}
q_i^-	0.082163	0.00009	-0.007494	0.082163	-0.000033	-0.007494
q_i^+	0.086067	0.000010	-0.007151	0.086067	-0.000026	-0.007151

Optimization methods

$$u_{i\alpha}^{-} \Leftarrow \begin{cases} \min u_i \\ \mathbf{L}(\mathbf{u},\mathbf{h}) = \mathbf{f}(\mathbf{h}) \\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha \end{cases} \qquad u_{i\alpha}^{+} \Leftarrow \begin{cases} \max u_i \\ \mathbf{L}(\mathbf{u},\mathbf{h}) = \mathbf{f}(\mathbf{h}) \\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha \end{cases}$$

$$u_{i\alpha}^{-} \Leftarrow \begin{cases} \min u_i \\ \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha \end{cases} \qquad u_{i\alpha}^{+} \Leftarrow \begin{cases} \max u_i \\ \mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}) \\ \mathbf{h} \in \hat{\mathbf{h}}_\alpha \end{cases}$$

These methods can be applied
to the very wide intervals

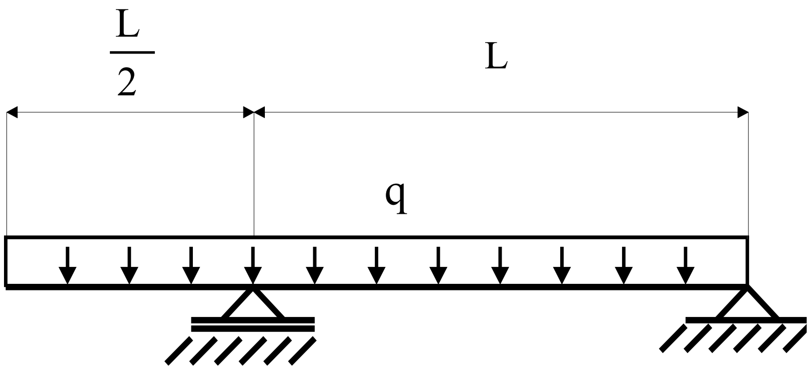
$$\hat{\mathbf{h}}_{\alpha}.$$

Function

$$\mathbf{u} = \mathbf{u}(\mathbf{h})$$

doesn't have to be monotone.

Numerical example



$$\begin{cases} \dfrac{d^2}{dx^2}\left(EJ\,\dfrac{d^2u}{dx^2} \right) = q(x), \\ u\left(\dfrac{L}{2}\right) = 0, \quad u\left(\dfrac{3L}{2}\right) = 0, \quad \dfrac{d^2u(0)}{dx^2} = 0, \quad \dfrac{d^2}{dx^2}u\left(\dfrac{3L}{2}\right) = 0 \end{cases}$$

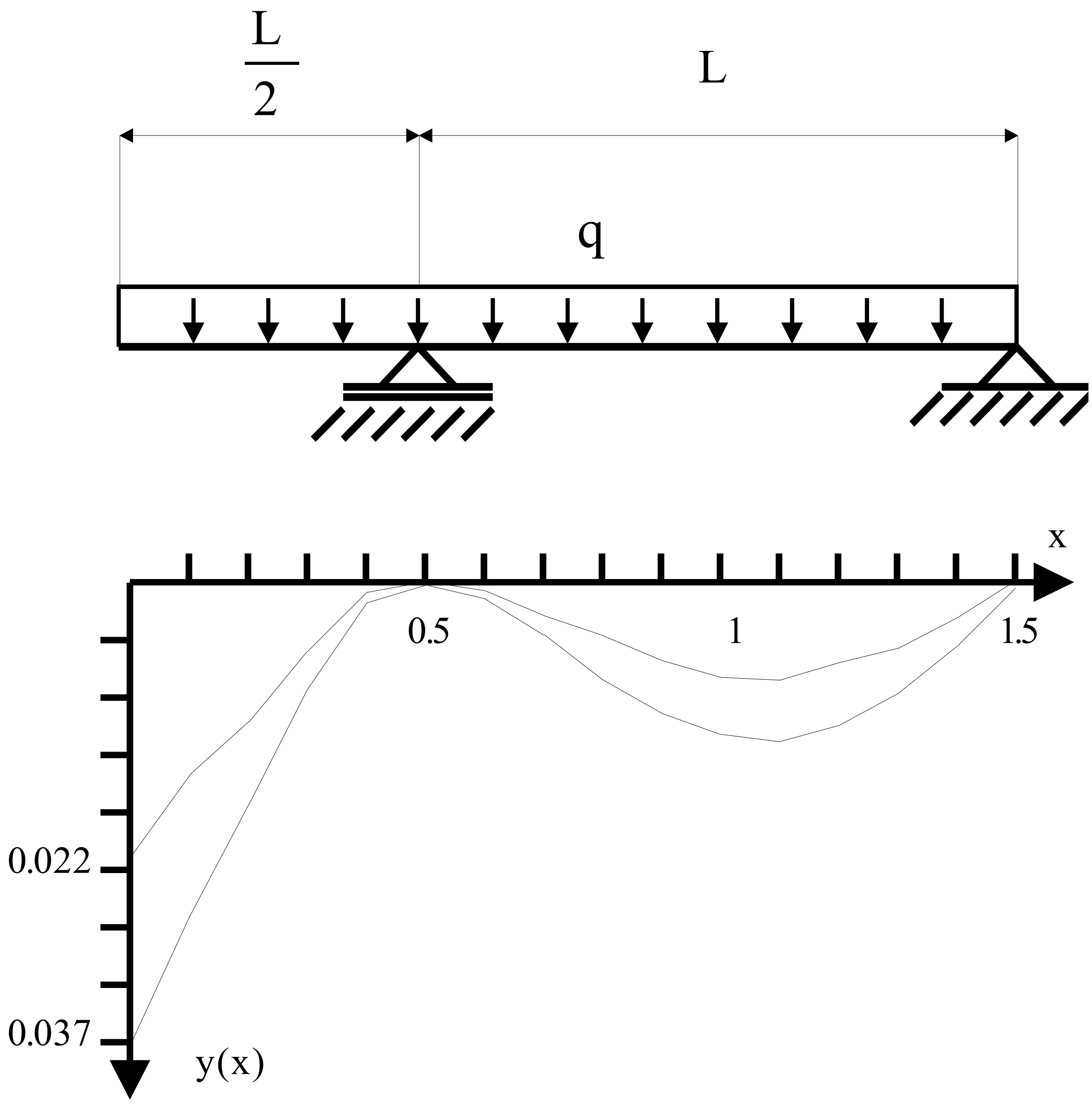
Numerical data

$$E \in [2 \cdot 10^5, 2.2 \cdot 10^5][MPa],\ J \in \left[\frac{0.049^4}{12}, \frac{0.051^4}{12}\right][m^4],$$

$$L \in [0.999, 1.001][m],\ q \in [9.9, 10.1][kN].$$

Analytical solution

$$u(x) = \begin{cases} \frac{1}{EJ}\left(\frac{1}{24}qx^4 - \frac{ql^3}{48}x + \frac{ql^4}{128}\right) \text{dla } x \in \left[0, \frac{L}{2}\right] \\ \frac{1}{\mathrm{EJ}}\left(\frac{1}{24}qx^4 - \frac{9}{48}qL\left(x - \frac{L}{2}\right)^3 - \frac{qL^3}{48}x + \frac{qL^4}{128}\right) \text{dla } x \in \left[\frac{L}{2}, \frac{3L}{2}\right] \end{cases}$$


L
2
L
q
x
0.5
1
1.5
0.022
0.037
y(x)

Other methods and applications

Iterative methods

Popova, E. D., On the Solution of Parametrised Linear Systems. In: W. Kraemer, J. Wolff von Gudenberg (Eds.): Scientific Computing, Validated Numerics, Interval Methods. Kluwer Acad. Publishers, 2001, pp. 127-138.

Muhanna L.R., Mullen L.R., Uncertainty in Mechanics. Problems - Interval Based - Approach. Journal of Engineering Mechanics, Vol. 127, No.6, 2002, pp.557-566

$$\mathbf{K}(\mathbf{h})\mathbf{u} = \mathbf{Q}(\mathbf{h}), \quad \mathbf{h} \in \hat{\mathbf{h}}_{\alpha}$$

$$K_{ij}(\mathbf{h}) = C_{ij}^{k} h_k$$

$$Q_j(\mathbf{h}) = C_j^{k} h_k$$

Inner solution Outer solution

$$\hat{\mathbf{u}}_{INNER}^{(i)} \subseteq \hat{\mathbf{u}} \subseteq \hat{\mathbf{u}}_{OUT}^{(i)}$$

$$\hat{\mathbf{u}}_{OUT}^{(i)} \to \hat{\mathbf{u}}$$

Valliappan S., Pham T.D., 1993,
Fuzzy Finite Element Analysis
of a Foundation on Elastic Soil Medium.
International Journal for Numerical
and Analytical Methods in Geomechanics, Vol.17, s.771-789

In some cases we can prove,
that the solution can be calculated using only
endpoints of given intervals.

The authors were solved some special
fuzzy partial differential equations
using only endpoints of given intervals.

Load combinations in civil engineering

Many existing civil engineering programs
can calculate extreme solutions
of partial differential equations
with interval parameters (only loads) e.g:
- ROBOT (http://www.robobat.com.pl/),
- CivilFEM (www.ingeciber.com).

These programs calculate
all possible combinations
and then calculate the extreme solutions
(some forces exclude each other).

Fuzzy eigenvalue problem

$$det\left(\mathbf{M}(\mathbf{h})-\lambda\mathbf{K}(\mathbf{h})\right)=0$$

$$\lambda_{\alpha}^{(i)}=\{\lambda^{(i)}:\det(\mathbf{M}(\mathbf{h})-\lambda\mathbf{K}(\mathbf{h})),\mathbf{h}\in\hat{\mathbf{h}}_{\alpha}\}$$

$$\mu(\lambda\,|\,\lambda_{F}^{(i)})=sup\{\alpha:\lambda\in\lambda_{\alpha}^{(i)}\}$$

Upper probability of the stability

$$Pl\{\mathrm{Re}(\lambda^{(i)}) < 0\} = \sup_{\lambda:\lambda<0} \mu(\lambda \,|\, \lambda_F^{(i)})$$

Random set Monte Carlo simulations

$$Pl(u_0 \in u(\hat{H})) = P_{\Omega}\{\omega : u \in u(\hat{H}_{\Omega}(\omega))\}$$

In some cases we cannot apply
fuzzy sets theory to solution of this problem.

Conclusions

Conclusions

1) Calculation of the solutions
of fuzzy partial differential equations
is in general very difficult (NP-hard).

2) In engineering applications the relation
between the solution and uncertain parameters
is usually monotone.

3) Using methods which are based on sensitivity
analysis we can solve very complicated problems
of computational mechanics.
(thousands degree of freedom)

4) If we apply the point monotonicity tests
we can use results which was generated
by the existing engineering software.

5) Reliable methods of solution
of fuzzy partial differential equations
are based on the interval arithmetic.
These methods have
high computational complexity.

6) In some cases
(e.g. if we know analytical solution)
optimization method can be applied.

7) In some special cases we can predict
the solution of fuzzy partial differential equations.

8) Fuzzy partial differential equation can be applied
to modeling of mechanical systems (structures)
with uncertain parameters.