Numerical solution of FEM equations with uncertain functional parameters

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Extreme values of monotone function

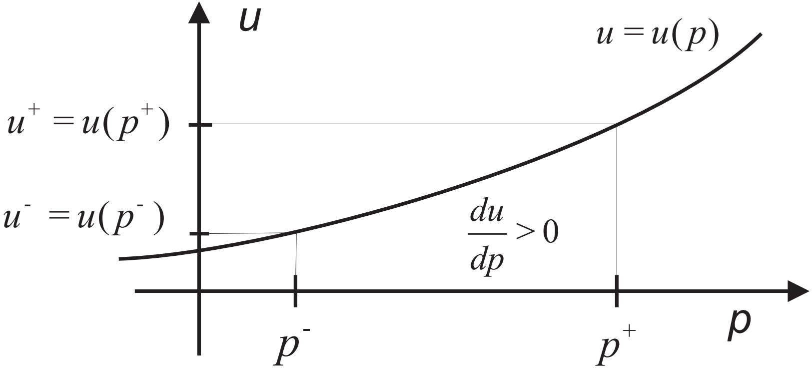


Figure: Extreme values of a monotone function $u = u(p)$ can be calculated by using upper and lower bounds of the parameters i.e. $p^-, p^+ \in R$.

Sensitivity analysis

$$u = u(p), \quad p \in [p^-, p^+] \tag{1}$$

$$\frac{du(p)}{dp} \geqslant 0, \quad for \quad p \in [p^-, p^+] \tag{2}$$

$$u^- = u(p^-), \quad u^+ = u(p^+) \tag{3}$$

$$\frac{du(p)}{dp} < 0, \quad for \quad p \in [p^-, p^+] \tag{4}$$

$$u^- = u(p^+), \quad u^+ = u(p^-) \tag{5}$$

Sensitivity analysis

$$\text{If} \quad \frac{du(p)}{dp} \geqslant 0, p_s^- = p^-, p_s^+ = p^+ \tag{6}$$

$$\text{If} \quad \frac{du(p)}{dp} < 0, p_s^- = p^+, p_s^+ = p^- \tag{7}$$

$$\boxed{u^- = u(p_s^-), \quad u^+ = u(p_s^+)} \tag{8}$$

Sensitivity analysis: functional parameters case

$$p = p(y) \in [p^-(y), p^+(y)] \tag{9}$$

$$u(x,p) = \int_0^L L(x,p(y))dy \approx \sum_i L(x,p(y_i))\Delta y_i \tag{10}$$

$$\frac{\partial u(x,p)}{\partial p(y_i)} \approx \frac{\partial L(x,p(y_i))}{\partial p(y_i)}\Delta y_i \tag{11}$$

$$\delta u(x,p) = \delta \int_0^L L(x,p(y))dy = \int_0^L \frac{\delta L(x,p(y))}{\delta p(y)}\delta p(y)dy \tag{12}$$

$$\frac{\delta u(x,p)}{\delta p(y)} = \frac{\partial L(x,p(y))}{\partial p(y)} \tag{13}$$

Sensitivity analysis: functional parameters case

$$p(y) \in [p^-(y), p^+(y)] \tag{14}$$

$$if \quad \frac{\delta u(x,p)}{\delta p(y)} \geqslant 0 \quad then \quad , \tag{15}$$

$$p_s^-(y) = p^-(y), \quad p_s^+(y) = p^+(y) \tag{16}$$

$$if \quad \frac{\delta u(x,p)}{\delta p(y)} < 0 \quad then \quad , \tag{17}$$

$$p_s^-(y) = p^+(y), \quad p_s^+(y) = p^-(y) \tag{18}$$

$$\boxed{u^-(x) = u(x, p_s^-), \quad u^+(x) = u(x, p_s^+)} \tag{19}$$

Remarks

$$\boxed{u^{-}(x) = u(x, p_s^{-}), \quad u^{+}(x) = u(x, p_s^{+})} \tag{20}$$

In multidimensional case the set

$$\tilde{p} = \{(p_1, ..., p_m) : p_i \in [p_i^{-}(y), p_i^{+}(y)], y \in \Omega\} \tag{21}$$

may be very complicated.

Uncertainty in mechanics

Figure: Material properties and geometrical parameters of damaged structures

Uncertainty in mechanics

Figure: Material and geometrical properties of rocks

Uncertainty in mechanics

Figure: Material and geometrical properties of soil

Random variables

- Definition $X : \Omega \ni \omega \to X(\omega) \in R$
- Probability density function $P\{a \leq X \leq b\} = \int_a^b f(x)dx$

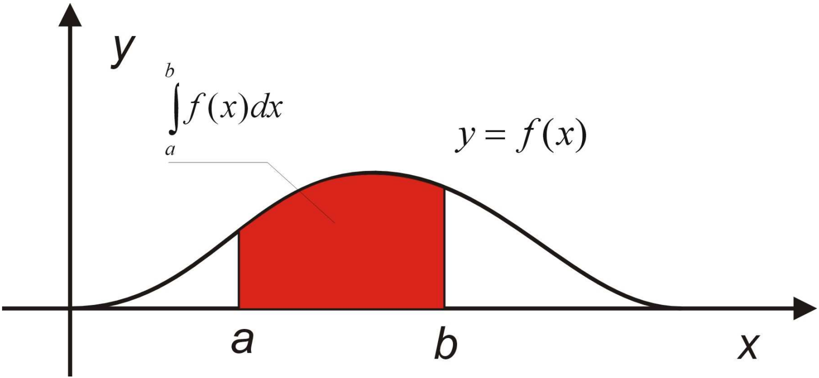


Figure: Probability that $P\{a \leq X \leq b\} = \int_a^b f(x)dx$

Application of random variables

- Random material characteristics e.g.
  - Young modulus $E : \Omega \ni \omega \to E(\omega) \in R$
  - Poisson number $\nu : \Omega \ni \omega \to \nu(\omega) \in R$
  - Random point load $P : \Omega \ni \omega \to P(\omega) \in R$
  - Random distributed load $q : \Omega \ni \omega \to q(\omega) \in R$
  - etc.
- Random parameters are characterised by using probability density function

$$P\{\omega : E_1 \leq E(\omega) \leq E_2\} = \int_{E_1}^{E_2} f_E(E)dE$$

Beam with random parameters

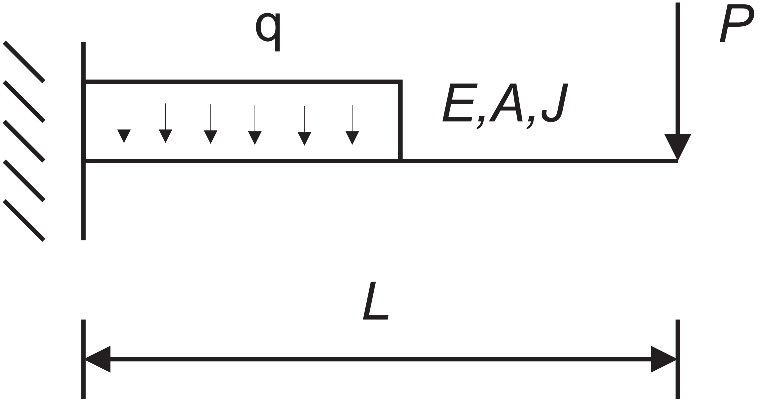


Figure: Beam with random parameters

Distributed load as a random variable

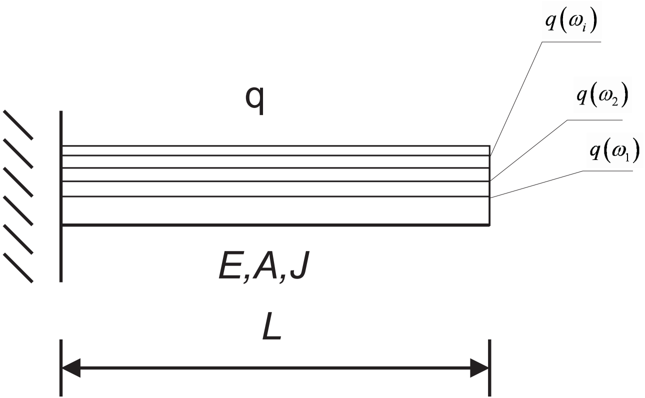


Figure: Beam with random distributed load

Distributed load as a random field

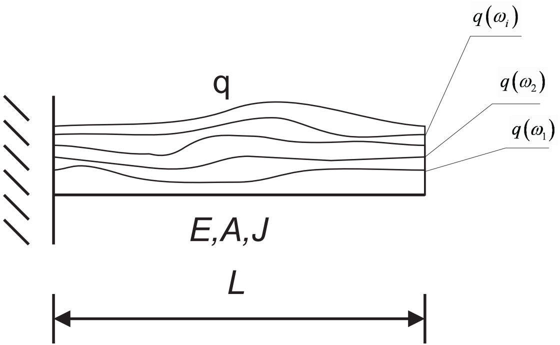
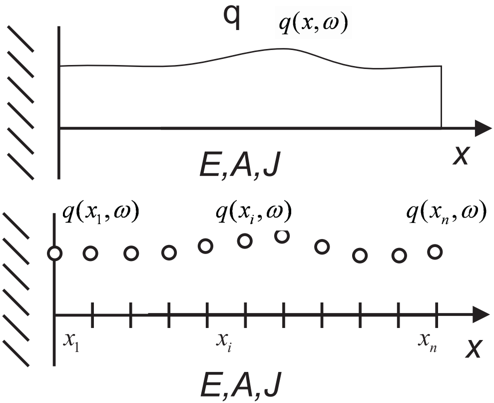


Figure: Beam with random distributed load

At this moment interval methods are not able to take into account more complicated types of dependency.

Discretization of random fields

- Definition $q : (\Omega, R) \ni (\omega, x) \rightarrow q(\omega, x) \in R$.
- Random fields can be approximated by the random vectors.
  $\{(q(x,\omega), x, \omega) : x \in [0, L], \omega \in \Omega\} \approx$
  $\{q(x_1,\omega), q(x_2,\omega), ..., q(x_n,\omega)\}$



Description of random vectors (discretized random fields)

For gaussian random fields we can describe the probability density function of the random process as multivariate normal distribution

$$f(x) = \frac{1}{(2\pi)^{\frac{N}{2}}|\Sigma|} exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right) \tag{22}$$

where

$$\mu = E[X] \tag{23}$$

$$\Sigma = E[(X - E[X])(X - E[X])^T] \tag{24}$$

Applications:
Stochastic ODE, PDE, stochastic FEM, FORM, SORM, Monte-Carlo methods etc.

Main problem:
How to get probabilistic characteristics (e.g. μ, Σ)?

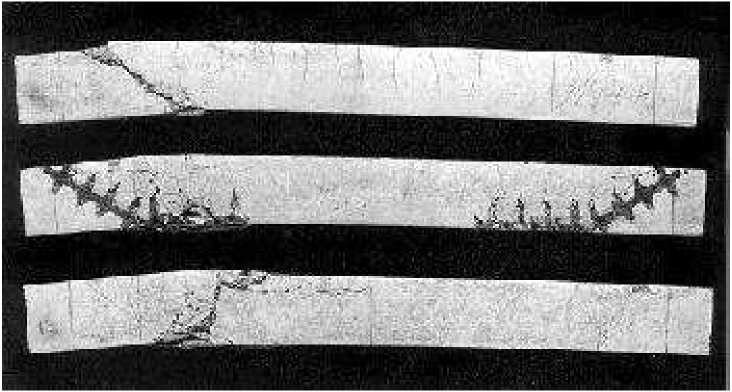


Figure: Concrete beams with cracks

Limitations of probabilistic methods

Elishakoff I., Possible Limitations of Probabilistic Methods in Engineering, ASME. Applied Mechanics Reviews, Vol.53, pp 19-36, 2000

- Lack of probabilistic data (because there is no time and money for collecting that data).
- Controversy related to likelihood interpretation of reliability and safety.
- Some researchers claim that probability doesn't exist.
- In many cases the problems are unique (particularly civil engineering applications) and it is hard to get reliable probabilistic data.
- In some cases data are unavailable because it is very hard to get the information about the values of particular parameter (e.g. material parameters of soil 2000 m under ground level).
- etc.

Safety factors

- Semi-probabilistic methods. Reliability index

$$\beta = -\Phi^{-1}(P_f) \tag{25}$$

Calibration of partial safety factors

$$\min_{\gamma} W(\gamma, \beta) \tag{26}$$

where W is some penalty function.

- Non-probabilistic definition

$$\gamma = \frac{x^{max}}{x^{design}} \tag{27}$$

where x^{design} is a design value, x^{max} is characteristic value.

Existing methods of modelling uncertainty
Design of structures using standard codes

Limit state design (Eurocode)

$$\frac{R_k}{\gamma_M} \geqslant E_d \tag{28}$$

where

$$E_d = \sum_{j \geqslant 1} \gamma_{Gj} G_j + \gamma_P P + \gamma_{Q1} Q_{k1} + \sum_{i \geqslant 1} \gamma_{Qi} \psi_{0i} Q_{ki} \tag{29}$$

R_k- is the characteristic value of the resistance
E_d-is the design value of the action effects
G_k-is the characteristic value of the permanent effects
P-it the characteristic value of prestressing
Q_k-is the characteristic value of the time variant actions
$\gamma_M, \gamma_{Gj}, \gamma_P, \gamma_{Q1}, \gamma_{Qi}$ - safety factors

In limit state design we have to predict worst case (worst case design)

- In existing codes only extreme load combinations have to be taken into account.

$$\forall P \in \{P_1, ..., P_N\}, \frac{R_k}{\gamma_M} \geqslant E_d(P) \tag{30}$$

- However in reality it will be better to include also variations loads, material and geometric parameters simultaneously

$$\forall p \in \tilde{p}, \frac{R_k}{\gamma_M} \geqslant E_d(p) \tag{31}$$

where p is a vector of all parameters. In general we have

$$\forall p \in \tilde{p}, g(p) \geqslant 0 \tag{32}$$

where g is any limit state function.

Simplest case of worst case analysis: interval parameters

- $\tilde{p} = [p^-, p^+]$ or $\tilde{p} = [p_1^-, p_1^+] \times [p_2^-, p_2^+] \times ... \times [p_m^-, p_m^+]$.
- Solution set of equations with interval parameters

$$u(\tilde{p}) = \{u : F(u, p) = Q(p), p \in \tilde{p}\} \qquad (33)$$

or

$$\Diamond u(\tilde{p}) = \Diamond\{u : F(u, p) = Q(p), p \in \tilde{p}\} \qquad (34)$$

where $\Diamond u(\tilde{p})$ is the smallest set which contain the set $u(\tilde{p})$. Above definition us valid also in the case of differential and integral equation.

- In particular case we have system of linear equation with interval parameters.

$$\Diamond u(\tilde{p}) = \Diamond\{u : K(p)u = Q(p), p \in \tilde{p}\} \qquad (35)$$

Convex model of uncertainty

- Ben-Haim, Y., and Elishakoff, I. (1990). Convex models of uncertainty in applied mechanics, Elsevier, New York.
- Ellipsoidal uncertainty

$$\tilde{p} = \left\{ (p_1, p_2) : \frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} \leqslant 1 \right\} \tag{36}$$

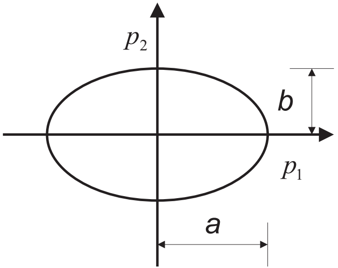


Figure: Ellipsoidal uncertainty

General set-valued uncertainty

- Let us consider equilibrium equation of beam under tension-compression.

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + n = 0 \tag{37}$$

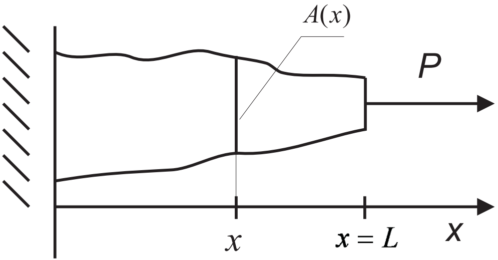


Figure: Rod under tension

General set-valued uncertainty

- Uncertain Young modulus

$$E(x) \in \tilde{E}(x) = [E^-(x), E^+(x)] \quad (38)$$

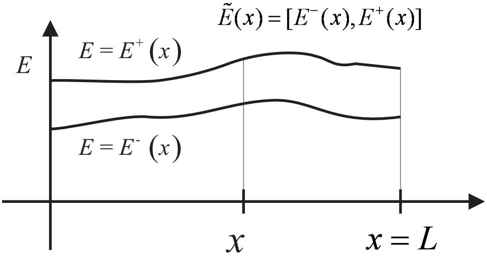


Figure: Set-valued Young modulus

Solution of equation with set valued Young modulus

- Solution at point x (displacements)

$$u(x) \in \tilde{u}(x) = [u^-(x), u^+(x)] = \Diamond\{u(x, E) : E \in \tilde{E}\} \quad (39)$$

- Solution of discretized equations in the nodal points

$$u \in \Diamond u(\tilde{E}_f) = \Diamond\{u(E) : E \in \tilde{E}\} \quad (40)$$

In this case E is a function $E : [0, L] \ni x \rightarrow E(x) \in R$ and $\tilde{E}$ is a functional space of functions form the interval $[0, L]$ to R where.

$$\forall E \in \tilde{E}, \forall x \in [0, L], E^-(x) \leqslant E(x) \leqslant E^+(x) \quad (41)$$

Solution of interval equation using endpoint combination method

- Let us consider interval equation

$$f(u,p)=0, \quad \textit{or equivalently} \quad u=u(p) \tag{42}$$

- Additionally lets assume that $x=x(p)$ is monotone, then

$$u^{-}=min\{u(p^{-}),u(p^{+})\}, \quad x^{+}=max\{u(p^{-}),u(p^{+})\} \tag{43}$$

In multidimensional case in order to find the solution we have to solve 2^m (where m is a number of uncertain parameters).

$$u^{-}=\min\{u(p_1^{\pm},p_2^{\pm},...,p_m^{\pm})\} \tag{44}$$

$$u^{+}=\max\{u(p_1^{\pm},p_2^{\pm},...,p_m^{\pm})\} \tag{45}$$

Solution of interval equation using sensitivity analysis

- Let assume that the function $x = x(p)$ has positive derivative

$$\frac{du(p_0)}{dp} > 0 \tag{46}$$

$$u^- = u(p^-), \quad u^+ = u(p^+) \tag{47}$$

where

$$p_0 = mid(\tilde{p}) \tag{48}$$

Solution of interval equation using sensitivity analysis

- That algorithm can be also applied in multidimensional case

$$\frac{\partial u(p_0)}{\partial p_i} > 0 \tag{49}$$

For example sensitivity can be calculated in the following way

$$K(p_0)u(p_0) = Q(p_0) \tag{50}$$

$$K(p_0)\frac{\partial u(p_0)}{\partial p_i} = \frac{\partial Q(p_0)}{\partial p_i} - \frac{\partial K(p_0)}{\partial p_i}u(p_0) \tag{51}$$

Sensitivity of the solution of the differential equation of tension-compression problem

- That algorithm can be also applied in multidimensional case

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0 \tag{52}$$

$$u(0) = 0, \quad EA\frac{du(L)}{dx} = P \tag{53}$$

To the solution of that problem one can apply FEM method

$$u_h(x) = N_0(x)u_0 + N_1(x)u_1 \tag{54}$$

$$N_0(x) = \left(1 - \frac{x}{L}\right), \quad N_1(x) = \frac{x}{L} \tag{55}$$

where $u_0 = 0$.

Approximation of the value of integrals by a set of discrete values

$$K \cdot u = P \tag{56}$$

where

$$K = \int_{0}^{L} E(x)A(x)\frac{dN_1(x)}{dx}\frac{dN_1(x)}{dx}dx \tag{57}$$

and $u = u_1$.

$$\boxed{K \approx \sum_{i} E(x_i)A(x_i)\frac{dN_1(x_i)}{dx}\frac{dN_1(x_i)}{dx}\Delta x_i} \tag{58}$$

Sensitivity with respect to point values of Young modulus

$$\frac{\partial K}{\partial E(x_i)} \approx \frac{\partial}{\partial E(x_i)} \left(\sum_i E(x_i) A(x_i) \frac{dN_1(x_i)}{dx} \frac{dN_1(x_i)}{dx} \Delta x_i \right) \tag{59}$$

$$\boxed{\frac{\partial K}{\partial E(x_i)} \approx A(x_i) \frac{dN_1(x_i)}{dx} \frac{dN_1(x_i)}{dx} \Delta x_i = \frac{A(x_i)\Delta x_i}{L^2}} \tag{60}$$

Functional derivative

$$\frac{\delta K}{\delta\ E(x_i)} = \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i} \frac{\partial K}{\partial E(x_i)} \approx A(x_i) \frac{dN_1(x_i)}{dx} \frac{dN_1(x_i)}{dx} \tag{61}$$

Taylor expansion of the solution

Taylor Series

$$u(x, p_0 + \Delta p) \approx u(x, p_0) + \sum_i \frac{\partial u(x, p_0)}{\partial p(y_i)} \Delta p(y_i) + \dots \tag{62}$$

Functional Taylor Series

$$\boxed{u(x, p_0 + \delta p) \approx u(x, p_0) + \int_0^L \frac{\delta u(x, p_0)}{\delta p(y)} \delta p(y) dy + \dots} \tag{63}$$

Sensitivity analysis

Let's assume that

$$\frac{\delta u(x,p_0)}{\delta p(y)} \geqslant 0, \quad \textit{for} \quad y \in \tilde{y}_p^+ \subset [0,L] \tag{64}$$

$$\frac{\delta u(x,p_0)}{\delta p(y)} < 0, \quad \textit{for} \quad y \in \tilde{y}_p^- \subset [0,L] \tag{65}$$

additionally

$$p_s^+(y) = \begin{cases} p_0(y) + \delta p^+(y) & \textit{for} \quad y \in \tilde{y}_p^+ \\ p_0(y) + \delta p^-(y) & \textit{for} \quad y \in \tilde{y}_p^+ \end{cases} \tag{66}$$

$$p_s^-(y) = \begin{cases} p_0(y) + \delta p^-(y) & \textit{for} \quad y \in \tilde{y}_p^+ \\ p_0(y) + \delta p^+(y) & \textit{for} \quad y \in \tilde{y}_p^+ \end{cases} \tag{67}$$

then

$$u^-(x) = u(x, p_s^-), \quad u^+(x) = u(x, p_s^+) \tag{68}$$

Example: tension problem, sensitivity with respect to E

The sensitivity can be calculated form the equation

$$K(E_0)\frac{\delta u(p_0)}{\delta E(x_i)} = \frac{\delta Q(p_0)}{\delta E(x_i)} - \frac{\delta K(p_0)}{\delta E(x_i)} u(p_0) \tag{69}$$

The result of calculation is the following

$$\frac{\delta u(x, p_0)}{\delta E(y)} = -\frac{P}{E^2(y) \cdot A(y)} < 0, \quad for \quad y \in [0, L] \tag{70}$$

then

$$E_s^+(y) = E_0(y) + E^-(y) \tag{71}$$

$$E_s^-(y) = E_0(y) + E^+(y) \tag{72}$$

and extreme values can be calculated by using

$$u^-(x) = u(x, E_s^-), \quad u^+(x) = u(x, E_s^+) \tag{73}$$

Example: tension problem, sensitivity with respect to A and E

The result of calculation is the following

$$\frac{\delta u(x, p_0)}{\delta A(y)} = -\frac{P}{E(y) \cdot A^2(y)} < 0, \quad \textit{for} \quad y \in [0, L] \qquad (74)$$

then

$$A_s^+(y) = A_0(y) + A^-(y) \qquad (75)$$

$$A_s^-(y) = A_0(y) + A^+(y) \qquad (76)$$

and extreme values can be calculated by using

$$u^-(x) = u(x, A_s^-, E_s^-), \quad u^+(x) = u(x, A_s^+, E_s^+) \qquad (77)$$

Sensitivity analysis: general case

Let's assume that

$$\frac{\delta u(x, p_0)}{\delta p_i(y)} \geqslant 0, \quad \textit{for} \quad y \in \tilde{y}_p^+ \subset [0, L] \tag{78}$$

$$\frac{\delta u(x, p_0)}{\delta p_i(y)} < 0, \quad \textit{for} \quad y \in \tilde{y}_p^- \subset [0, L] \tag{79}$$

additionally

$$p_{si}^+(y) = \begin{cases} p_{0i}(y) + \delta p_i^+(y) & \textit{for} \quad y \in \tilde{y}_p^+ \\ p_{0i}(y) + \delta p_i^-(y) & \textit{for} \quad y \in \tilde{y}_p^+ \end{cases} \tag{80}$$

$$p_{si}^-(y) = \begin{cases} p_{0i}(y) + \delta p_i^-(y) & \textit{for} \quad y \in \tilde{y}_p^+ \\ p_{0i}(y) + \delta p_i^+(y) & \textit{for} \quad y \in \tilde{y}_p^+ \end{cases} \tag{81}$$

then

$$u^-(x) = u(x, p_{s1}^-, ...p_{sm}^-), \quad u^+(x) = u(x, p_{s1}^+, ..., p_{sm}^+) \tag{82}$$

Sensitivity analysis: general case

$$\boxed{u^-(x) = u(x, p_s^-), \quad u^+(x) = u(x, p_s^+)} \tag{83}$$

where

$$p_s^- = (p_{s1}^-, ...p_{sm}^-), \quad p_s^+ = (p_{s1}^+, ..., p_{sm}^+) \tag{84}$$

Numerical calculation of functional derivative

$$\frac{\delta u(x, p_0)}{\delta p_i(y)} \approx \frac{u(x, p_0 + \delta p_i(y)) - u(x, p_0)}{\Delta p_i(y)} \tag{85}$$

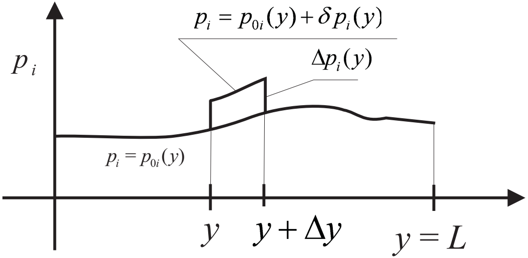


Figure: Function variation

Taylor expansion method - function parameter case

Taylor series (first order method)

$$u(x, p_0 + \Delta p) \approx u(x, p_0) + \sum_i \frac{\partial u(x, p_0)}{\partial p(y_i)} \Delta p(y_i) \tag{86}$$

$$u(x)^- \approx u(x, p_0) - \sum_i \left| \frac{\partial u(x, p_0)}{\partial p(y_i)} \right| |\Delta p(y_i)| \tag{87}$$

$$u(x)^+ \approx u(x, p_0) + \sum_i \left| \frac{\partial u(x, p_0)}{\partial p(y_i)} \right| |\Delta p(y_i)| \tag{88}$$

Taylor expansion method - function parameter case

Taylor series (first order method)

$$u(x, p_0 + \Delta p) \approx u(x, p_0) + \sum_j \sum_i \frac{\partial u(x, p_0)}{\partial p_j(y_i)} \Delta p_j(y_i) \qquad (89)$$

$$u(x)^- \approx u(x, p_0) - \sum_j \sum_i \left| \frac{\partial u(x, p_0)}{\partial p_j(y_i)} \right| |\Delta p_j(y_i)| \qquad (90)$$

$$u(x)^+ \approx u(x, p_0) + \sum_j \sum_i \left| \frac{\partial u(x, p_0)}{\partial p_j(y_i)} \right| |\Delta p_j(y_i)| \qquad (91)$$

Conclusions

- Using functional derivative it is possible to find solution of equation with uncertain functional parameters.
- The method can be applied to solution of large class of engineering problems with uncertain filed.
- The method can be applied to solution of linear and nonlinear problems of computational mechanics with uncertain filed.
- The algorithm of sensitivity analysis method method can be parallel.