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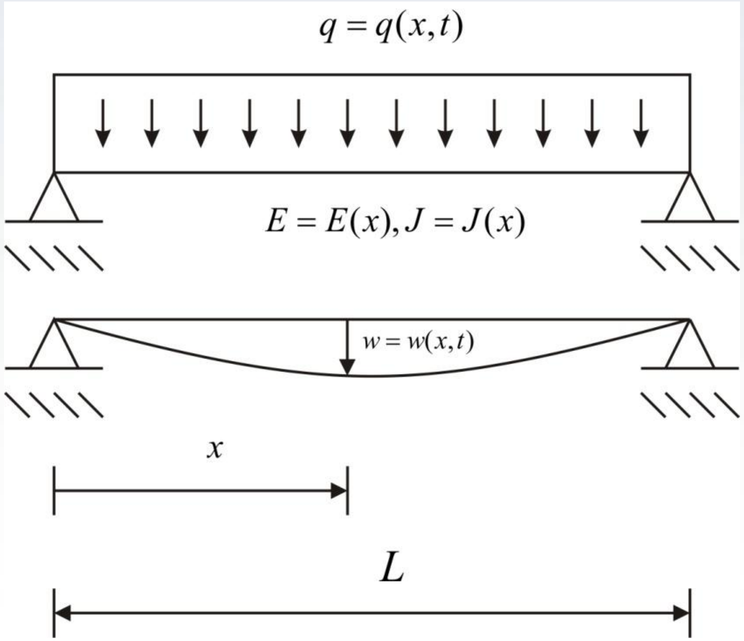
Dependency Problem in the Modeling of Beams with Uncertain Parameters

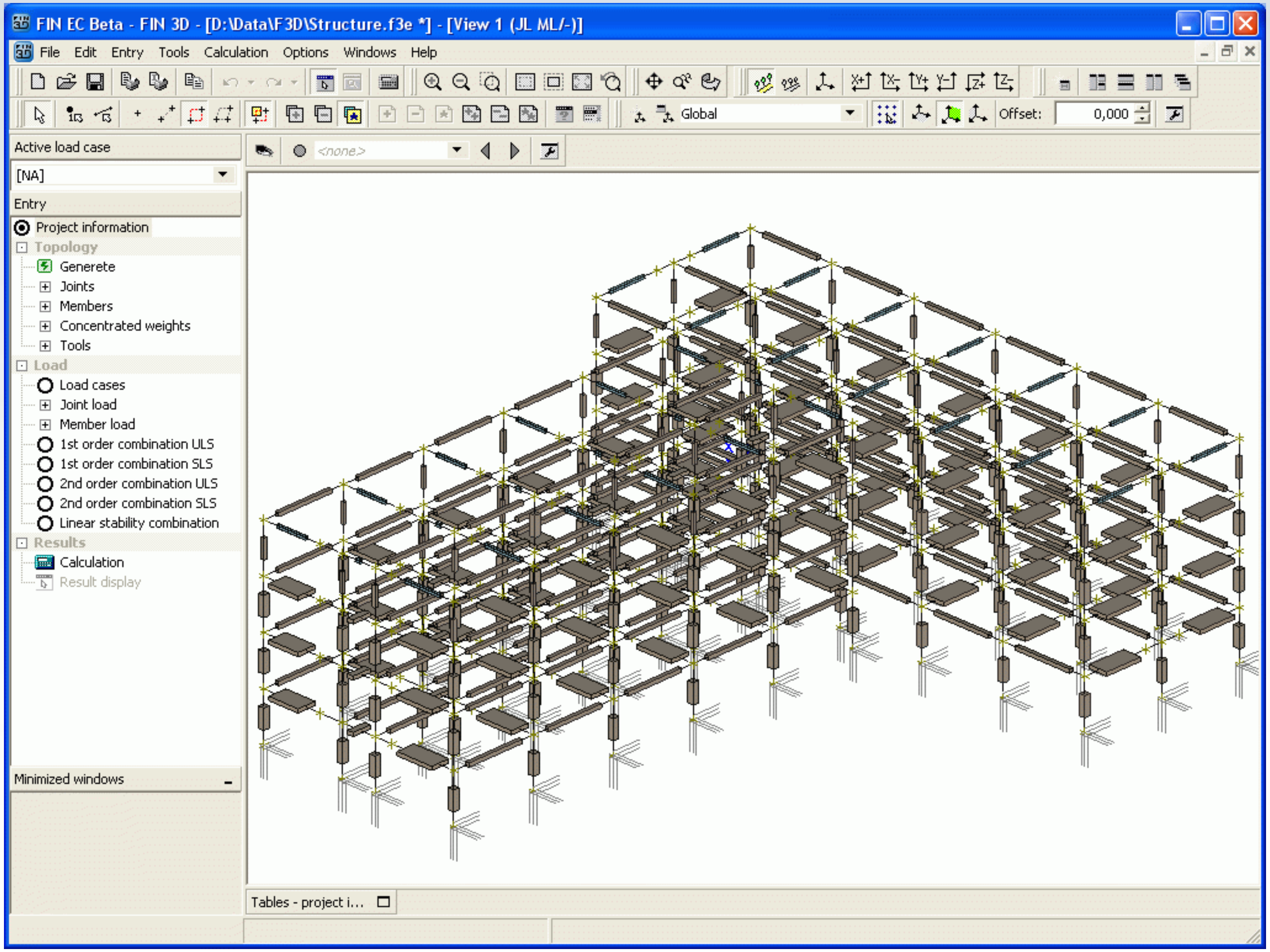
Frame structures

Beam equation

$$\frac{\partial^2}{\partial x^2}\left(EJ\,\frac{\partial^2 w}{\partial x^2}\right) = q - \rho A\,\frac{\partial^2 w}{\partial t^2}$$



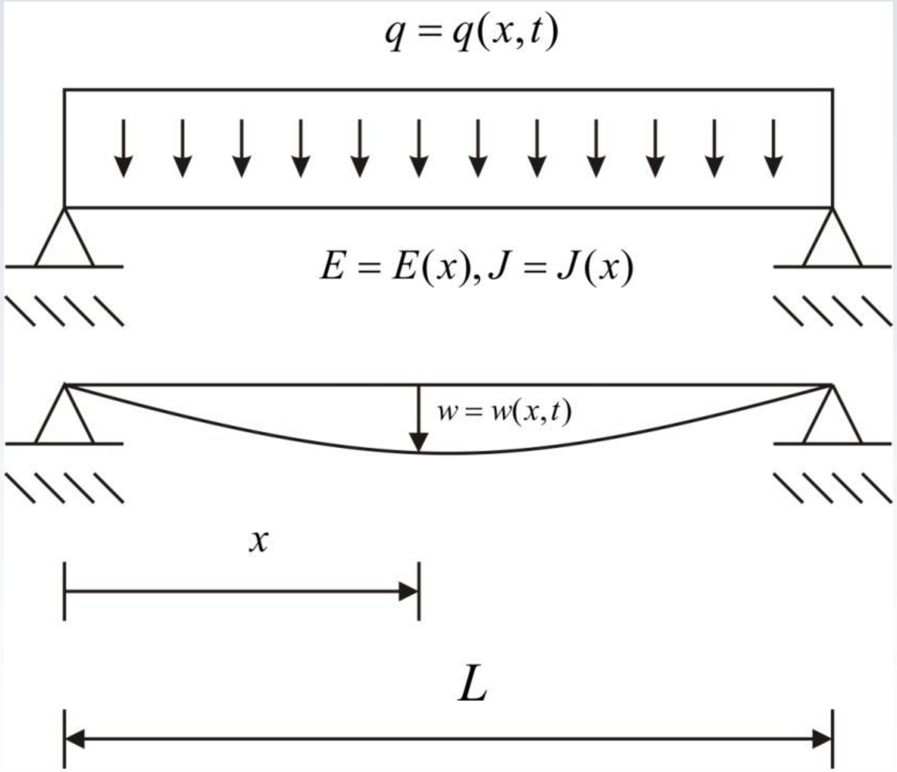
Simple problem …



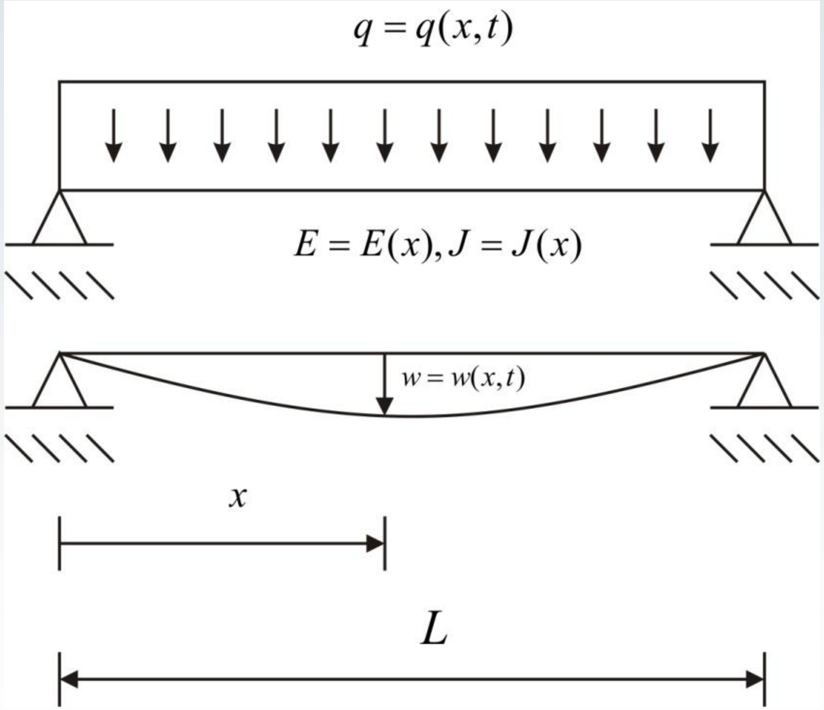
usually no dynamics …

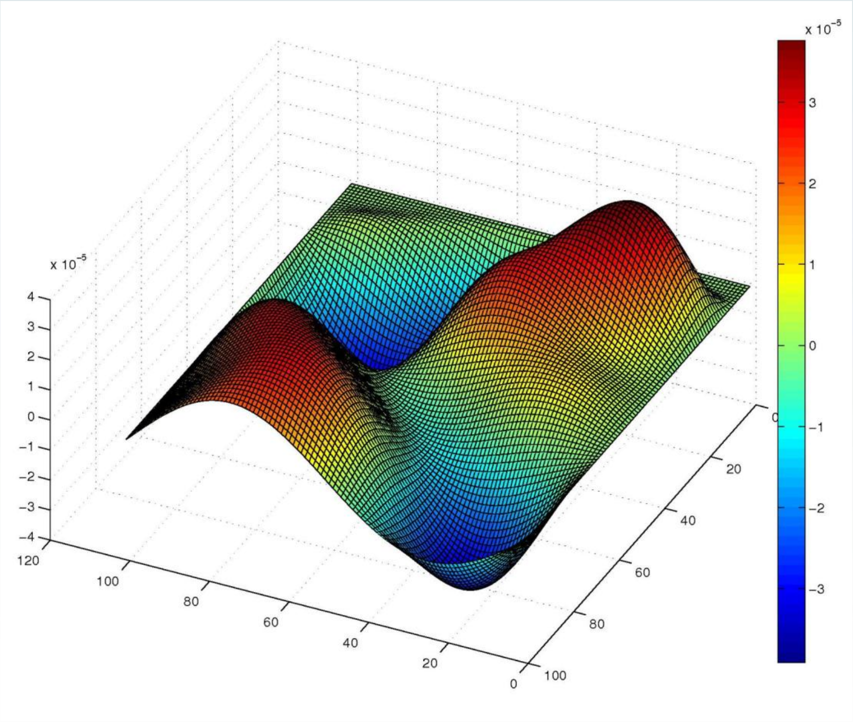
Boundary and initial conditions

$$
\begin{cases}
EJ\,\dfrac{\partial^4 w}{\partial x^4} = q - \rho A\,\dfrac{\partial^2 w}{\partial t^2} \\
w(0,t) = 0 \\
w(L,t) = 0 \\
\dfrac{\partial^2 w(0,t)}{\partial x^2} = 0 \\
\dfrac{\partial^2 w(L,t)}{\partial x^2} = 0 \\
w(x,0) = w_0(x) \\
v(x,0) = \dfrac{\partial w(x,0)}{\partial t} = v_0(x)
\end{cases}
$$



Example solution





Interval parameters

$$ax = b \Rightarrow x = \frac{b}{a}$$

$$2x = 4 \Rightarrow x = \frac{4}{2}$$

$$a \in \left[1,3\right] = \mathbf{a}$$

$$b \in \left[3,5\right] = \mathbf{b}$$

$$\mathbf{ax} = \mathbf{b} \Rightarrow \left[1,3\right]\mathbf{x} = \left[3,5\right]$$

$$\mathbf{x} = ?$$

United solution set

$$[1,2]x = [1,4]$$

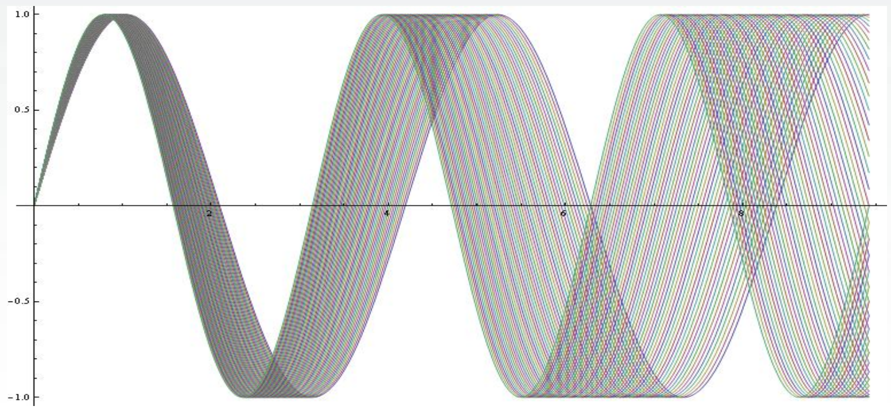
$$\mathbf{x} = \frac{[1,4]}{[1,2]} = \left[\frac{1}{2}, 4\right]$$

because

$$\mathbf{x} = \{x : ax = b, a \in [1,2], b \in [1,4]\}$$

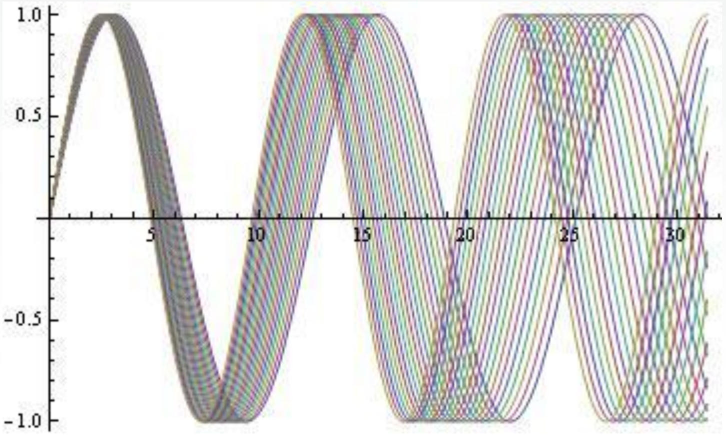
Interval equation

$$\begin{cases} y' = p\cos(pt) \\ y(0) = 0 \\ p \in \left[\underline{p}, \overline{p}\right] \end{cases} \quad \Longrightarrow \quad y = \sin(pt)$$

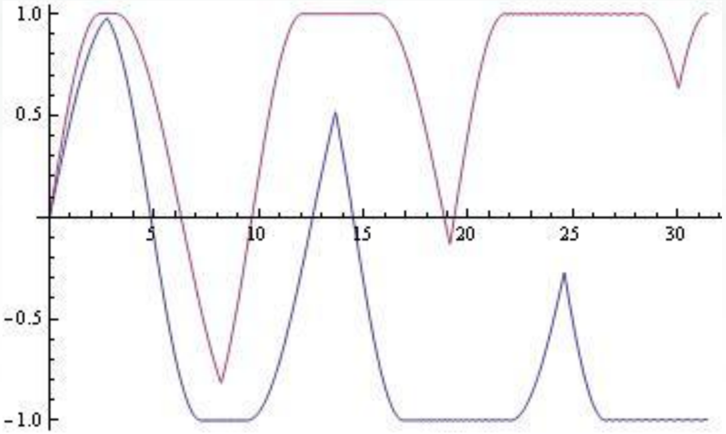


Interval solution

$$\underline{y}(t) = \min\{y(t,p) : p \in [\underline{p}, \overline{p}]\}$$
$$\overline{y}(t) = \max\{y(t,p) : p \in [\underline{p}, \overline{p}]\}$$



$\overline{y}(t)$



$\underline{y}(t)$

Implicit Finite Difference Method

$$
\begin{cases}
w(0,t) = 0 \\
\dfrac{\partial^2 w(0,t)}{dx^2} = 0 \\
EJ\,\dfrac{\partial^4 w}{\partial x^4} = q - \rho A\,\dfrac{\partial^2 w}{\partial t^2} \\
\dfrac{\partial^2 w(L,t)}{dx^2} = 0 \\
w(L,t) = 0 \\
\\
w(x,0) = w_0(x) \\
v(x,0) = \dfrac{\partial w(x,0)}{\partial t} = v_0(x)
\end{cases}
\Rightarrow
\begin{cases}
w_{0,j+1} = 0 \\
\dfrac{w_{0,j+1} - 2w_{1,j+1} + w_{2,j+1}}{\Delta x^2} = 0 \\
E_{i,j+1} J_{i,j+1}\,\dfrac{w_{i+2,j+1} - 4w_{i+1,j+1} + 6w_{i,j+1} - 4w_{i-1,j+1} + w_{i-2,j+1}}{\Delta x^4} = q_{i,j+1} - \rho_{i,j+1} A_{i,j+1}\,\dfrac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta t^2} \\
\dfrac{w_{n-2,j} - 2w_{n-1,j} + w_{n,j}}{\Delta x^2} = 0 \\
w_{n,j+1} = 0 \\
\\
w_{i,0} = w_i^* \\
\dfrac{w_{i,1} - w_{i,0}}{\Delta t} = v_i^*
\end{cases}
$$

Implicit Finite Difference Method

$$\frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4}w_{i+2,j+1} - 4\frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4}w_{i+1,j+1} + 6\frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4}w_{i,j+1} - 4\frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4}w_{i-1,j+1} + \frac{E_{i,j+1}J_{i,j+1}}{\Delta x^4}w_{i-2,j+1} + \frac{\rho_{i,j+1}A_{i,j+1}}{\Delta t^2}w_{i,j+1} =$$

$$= q_{i,j+1} + \rho_{i,j+1}A_{i,j+1}\frac{2w_{i,j} - w_{i,j-1}}{\Delta t^2}.$$

$$A(E^{j+1}, J^{j+1}, A^{j+1}, \rho^{j+1})w^{j+1} = b(q^{j+1}, \rho^{j+1}, A^{j+1}, w^j, w^{j-1})$$

$$\Downarrow$$

$$w^{j+1}$$

$$E^{j+1} = [E_{0,j+1}, E_{1,j+1}, \ldots, E_{n,j+1}], J^{j+1} = [J_{0,j+1}, J_{1,j+1}, \ldots, J_{n,j+1}], \ldots$$

Matrix A

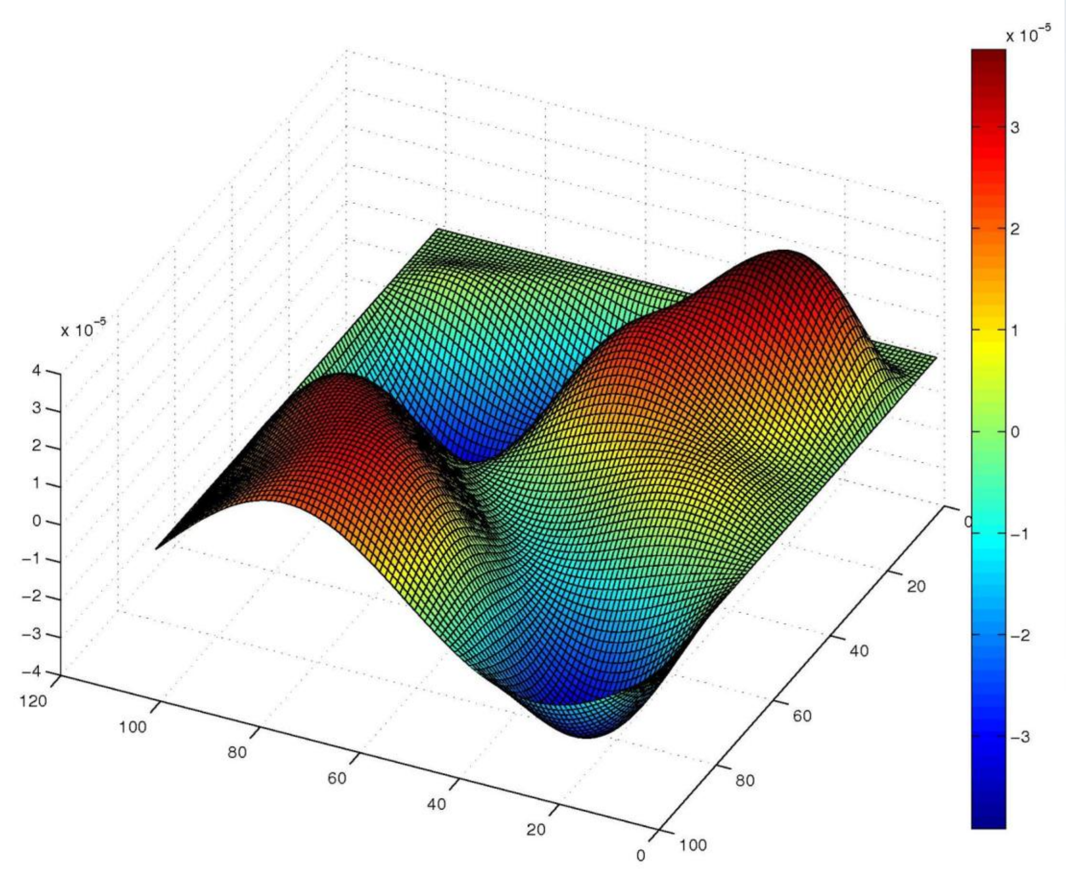
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \frac{EJ}{\Delta x^4} & \frac{-4EJ}{\Delta x^4} & \frac{6EJ}{\Delta x^4} + \frac{\rho A}{\Delta t^2} & \frac{-4EJ}{\Delta x^4} & \frac{EJ}{\Delta x^4} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{EJ}{\Delta x^4} & \frac{-4EJ}{\Delta x^4} & \frac{6EJ}{\Delta x^4} + \frac{\rho A}{\Delta t^2} & \frac{-4EJ}{\Delta x^4} & \frac{EJ}{\Delta x^4} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \frac{EJ}{\Delta x^4} & \frac{-4EJ}{\Delta x^4} & \frac{6EJ}{\Delta x^4} + \frac{\rho A}{\Delta t^2} & \frac{-4EJ}{\Delta x^4} & \frac{EJ}{\Delta x^4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Vector b

$$b = \begin{bmatrix} 0 \\ 0 \\ q_{2,j+1} + \frac{\rho A}{\Delta t^2}\left(2w_{2,j} - w_{2,j-1}\right) \\ q_{3,j+1} + \frac{\rho A}{\Delta t^2}\left(2w_{3,j} - w_{3,j-1}\right) \\ \cdots \\ q_{n-2,j+1} + \frac{\rho A}{\Delta t^2}\left(2w_{n-2,j} - w_{n-2,j-1}\right) \\ 0 \\ 0 \end{bmatrix}$$

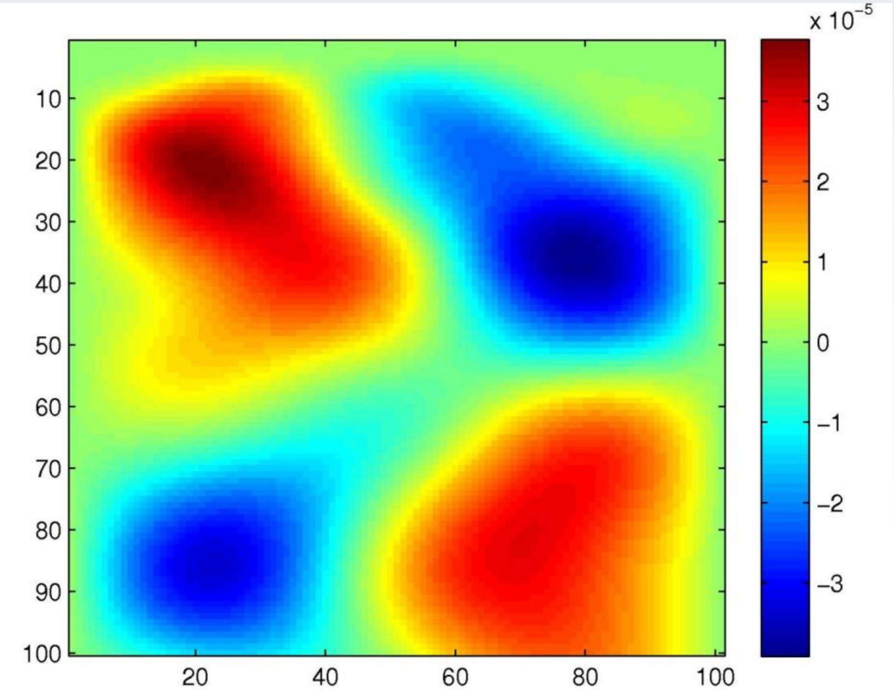
Example solution

$$w = w(x, t)$$



Example solution

$$w = w(x,t)$$



Dependency problem

- Parameter dependent system of equations

$$A(p)u = b(p), p \in \mathbf{p}$$

- Interval equation

$$[A]u = [b]$$

Dependency problem

$$Solution(A(p)u = b(p), p \in \mathbf{p})$$

$$\neq$$

$$Solution([A]u = [b])$$

Iterative solution

$$w_{i+1} = \Phi(w_i, w_{i-1}, p)$$

$$[w_{i+1}] = \Phi([w_i],[w_{i-1}], p)$$

Non-iterative formulation

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

$$y_{i+1} = y_i + y_i \Delta t$$

$$y_0 = 1$$
$$y_1 = y_0 + y_0 \Delta t$$
$$y_2 = y_1 + y_1 \Delta t$$
$$y_3 = y_2 + y_2 \Delta t$$
$$\ldots$$

Non-iterative formulation

$$\begin{cases} y_0 = 1 \\ y_1 - y_0 - y_0\Delta t = 0 \\ y_2 - y_1 - y_1\Delta t = 0 \\ y_3 - y_2 - y_2\Delta t = 0 \end{cases}$$

$$\begin{cases} y_0 = 1 \\ -\left(1 + \Delta t\right) y_0 + y_1 = 0 \\ -\left(1 + \Delta t\right) y_1 + y_2 = 0 \\ -\left(1 + \Delta t\right) y_2 + y_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -(1+\Delta t) & 1 & 0 & 0 \\ 0 & -(1+\Delta t) & 1 & 0 \\ 0 & 0 & -(1+\Delta t) & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Non-iterative formulation

$$
b^j = \begin{bmatrix} 0 \\ 0 \\ q_{2,j+1} \\ q_{3,j+1} \\ \dots \\ q_{n-2,j+1} \\ 0 \\ 0 \end{bmatrix} + \frac{\rho A}{\Delta t^2} \begin{bmatrix} 0 & 0 & & & & & 0 & 0 \\ 0 & 0 & & & & & 0 & 0 \\ & & 2 & & & & & \\ & & & 2 & & & & \\ & & & & \dots & & & \\ & & & & & 2 & & \\ 0 & 0 & & & & & 0 & \\ 0 & 0 & & & & & & 0 \end{bmatrix} \begin{bmatrix} w_{0,j} \\ w_{1,j} \\ w_{2,j} \\ w_{3,j} \\ \dots \\ w_{n-2,j} \\ w_{n-1,j} \\ w_{n,j} \end{bmatrix} - \frac{\rho A}{\Delta t^2} \begin{bmatrix} 0 & 0 & & & & & 0 & 0 \\ 0 & 0 & & & & & 0 & 0 \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \dots & & & \\ & & & & & 1 & & \\ 0 & 0 & & & & & 0 & \\ 0 & 0 & & & & & & 0 \end{bmatrix} \begin{bmatrix} w_{0,j-1} \\ w_{1,j-1} \\ w_{2,j-1} \\ w_{3,j-1} \\ \dots \\ w_{n-2,j-1} \\ w_{n-1,j-1} \\ w_{n,j-1} \end{bmatrix} = Q^j + M_2 w^j + M_1 w^{j-1}
$$

$$
M_2 = \frac{\rho A}{\Delta t^2} \begin{bmatrix} 0 & 0 & & & & & 0 & 0 \\ 0 & 0 & & & & & 0 & 0 \\ & & 2 & & & & & \\ & & & 2 & & & & \\ & & & & \dots & & & \\ & & & & & 2 & & \\ 0 & 0 & & & & & 0 & \\ 0 & 0 & & & & & & 0 \end{bmatrix}, M_1 = -\frac{\rho A}{\Delta t^2} \begin{bmatrix} 0 & 0 & & & & & 0 & 0 \\ 0 & 0 & & & & & 0 & 0 \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \dots & & & \\ & & & & & 1 & & \\ 0 & 0 & & & & & 0 & \\ 0 & 0 & & & & & & 0 \end{bmatrix}
$$

Non-iterative formulation

$$Aw^{j+1} = Q^j + M_2 w^j + M_1 w^{j-1}$$

$$w^0 = 0$$

$$w^1 = 0$$

$$Aw^2 = Q^1 + M_2 w^1 + M_1 w^0$$

$$Aw^3 = Q^2 + M_2 w^2 + M_1 w^1$$

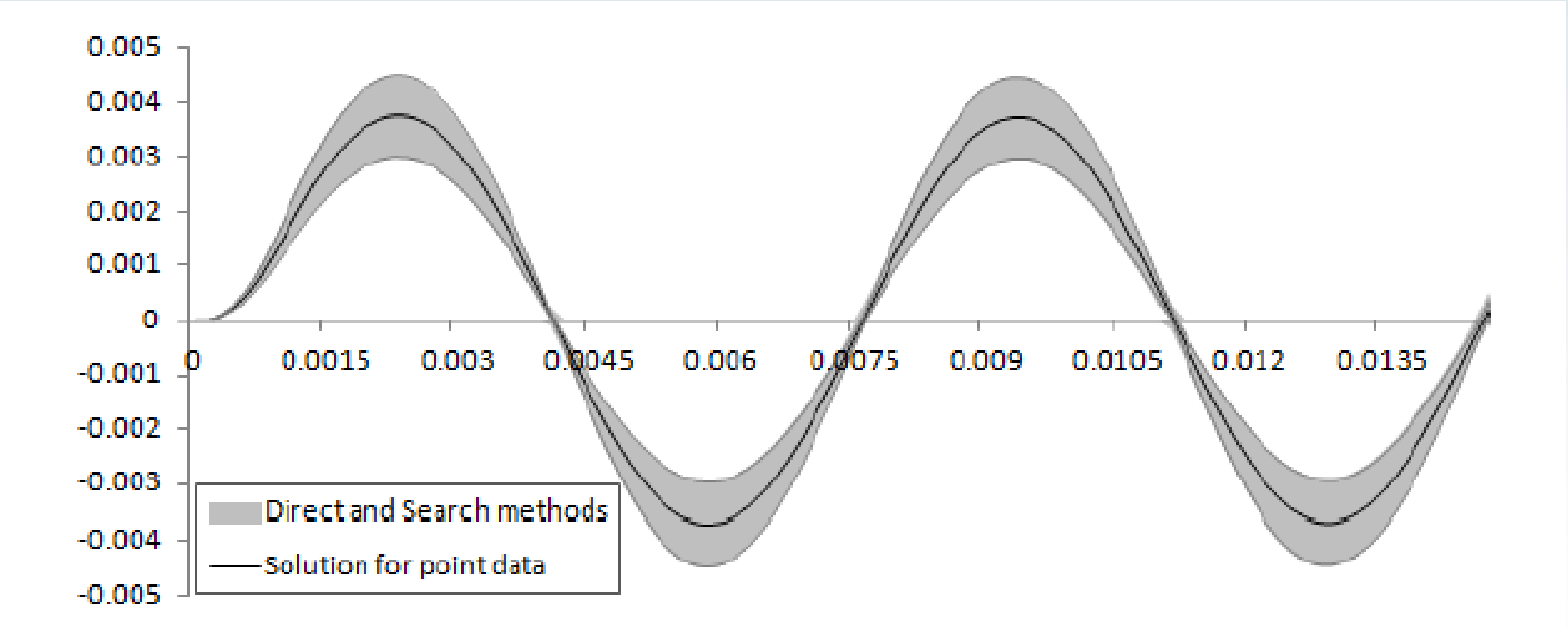
$$Aw^4 = Q^3 + M_2 w^3 + M_1 w^2$$

Non-iterative formulation

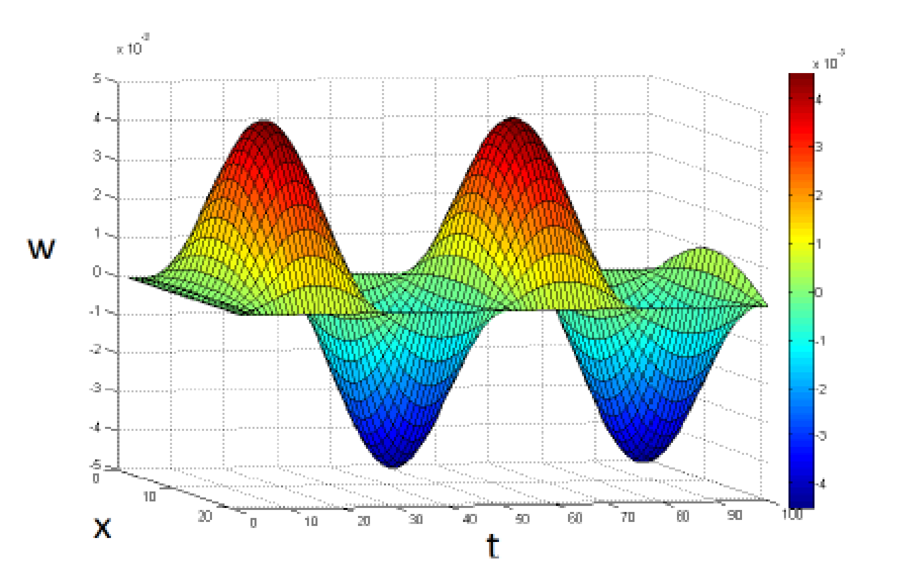
$$\begin{cases} w^0 = 0 \\ w^1 = 0 \\ Aw^2 - M_2 w^1 - M_1 w^0 = Q^1 \\ Aw^3 - M_2 w^2 - M_1 w^1 = Q^2 \\ Aw^4 - M_2 w^3 + M_1 w^2 = Q^3 \end{cases}$$

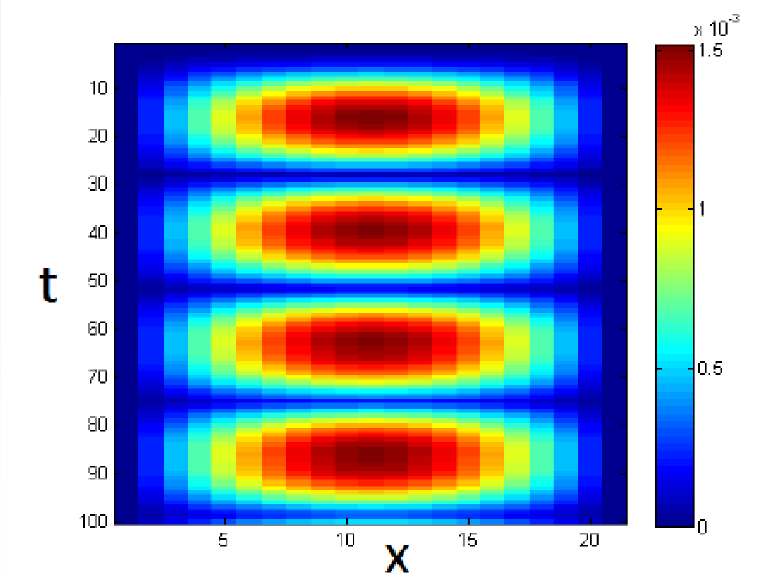
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -M_1 & -M_2 & A & 0 & 0 \\ 0 & -M_1 & -M_2 & A & 0 \\ 0 & 0 & -M_1 & -M_2 & A \end{bmatrix} \begin{bmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \\ w^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Q^1 \\ Q^2 \\ Q^3 \end{bmatrix}$$

Interval solution



Interval solution



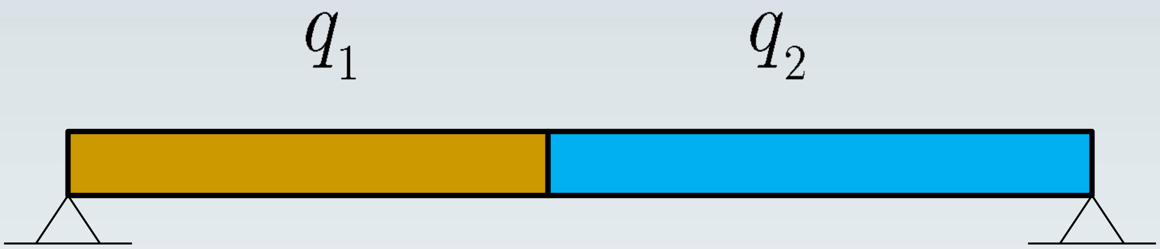


Dependency problem II

$$\begin{cases} EJ\frac{d^4w}{dx^4} = q \\ w(0) = 0 \\ w(L) = 0 \\ \frac{\partial^2 w(0)}{dx^2} = 0 \\ \frac{\partial^2 w(L)}{dx^2} = 0 \end{cases}$$

$$w_1(x, q, E) = \frac{q}{24EJ}\left(L^3 - 2Lx^2 - x^3\right) = \frac{q}{E}\varphi(x)$$

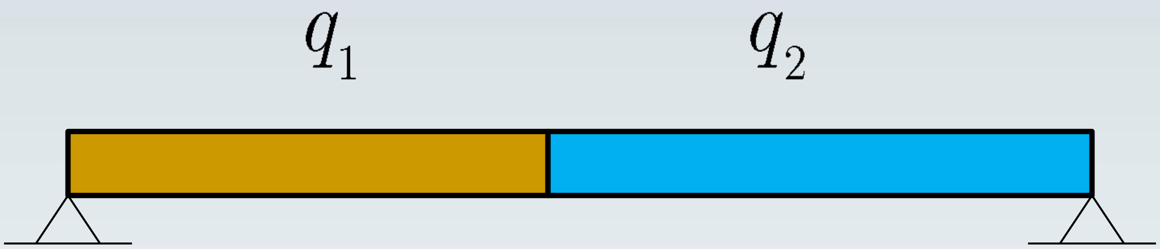
Dependency problem II



$$w_2(x, q_1, q_2, E) = (q_1\varphi_1(x) + q_2\varphi_2(x))\,\frac{1}{E}$$

$$w(x, q_1, ..., q_n, E) = (q_1\varphi_1(x) + ... + q_n\varphi_2(x))\,\frac{1}{E}$$

Dependency problem II



$$\underline{w}_1(x) = \underline{w}_2(x) = \ldots = \underline{w}_n(x)$$

$$\overline{w}_1(x) = \overline{w}_2(x) = \ldots = \overline{w}_n(x)$$

Functional derivative

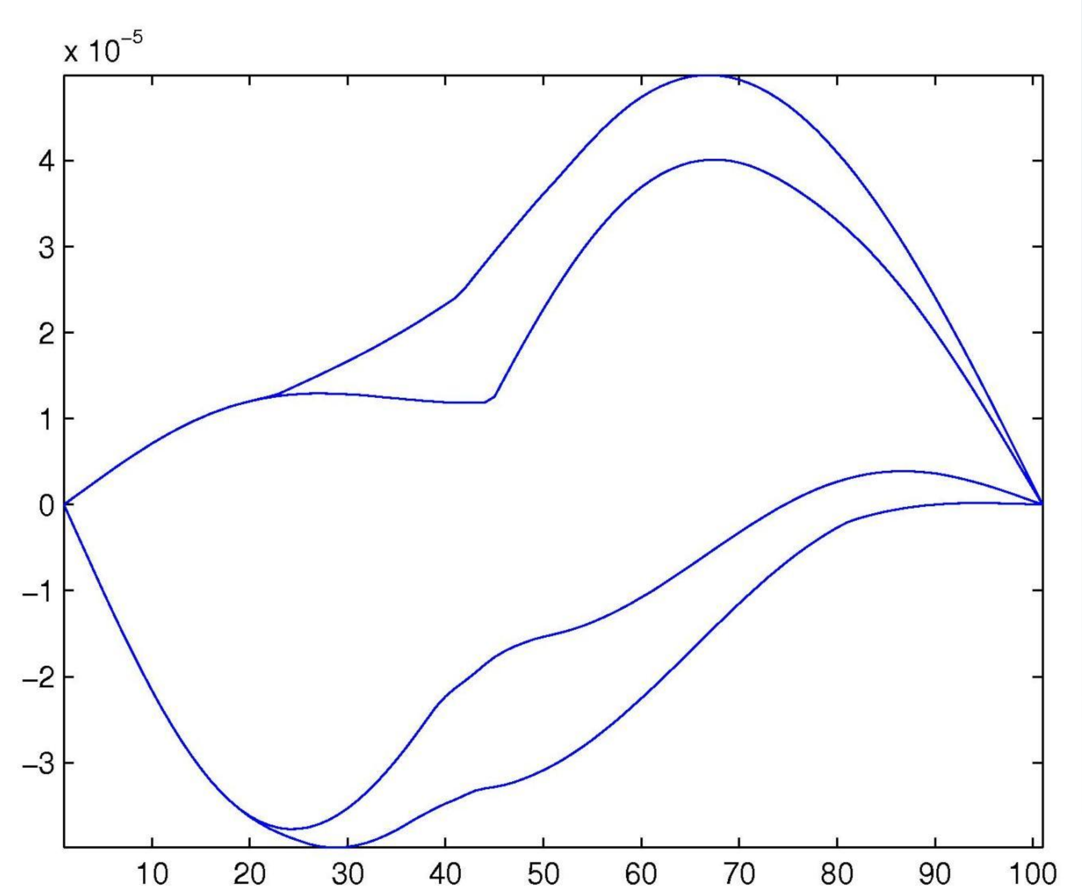
$$w(x) = -\int_0^x \frac{P}{\kappa^2 GA} d\xi + \int_0^x \left(\int_0^\eta \frac{P(\xi - L)}{EJ} d\xi \right) d\eta$$

$$\frac{\partial w(x,E)}{\partial E(\xi)} = \frac{P(L-\xi)}{E^2(\xi)J} \quad for \quad \xi \in [0,x]$$

Analytical solution of free vibrations

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4wL^4}{n^5 \pi^5 EI} \sin \frac{n\pi}{L} x \cos \omega t$$

Dependency problem II



Thank you