Calculation of reliability of structures by using interval probability

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Abstract

There are a lot of problems in engineering which are connected with the necessity not to neglect the influence of uncertainties in parameters (these are for example seismic vibrations, wind loaded structures and imperfection sensitive structures). Fuzzy numbers and possibility theory are used for problems where uncertainties in definition of input data do not allow for a treatment by means of probabilistic methods. Starting from scarce/uncertain body of information, fuzzy numbers are used to define possibility distributions as well as upper and lower bounds for a wide class of probability distributions compatible with available data. It is shown that fuzzy numbers give upper and lower bounds (with respect to probability) for value of the reliability of structures. Calculations were done using a special algorithm based on interval arithmetic.

Keywords :genetic algorithms, artificial neural networks, expert systems

1. Introduction

All parameters in mechanical systems are known with some accuracy [1]. When we have precise information about quality of them a deterministic analysis should be applied. Alternatively, when we have random characteristic of parameters, probabilistic methods should be applied. The situation is different if we do not possess a sufficient probabilistic characteristic information. Often only a very limited knowledge is available and appropriate mathematical tools are needed. When only extreme values of the parameters are known. In this circumstance one should apply the new analysis: convex modelling [2,8].

2. Relation between random sets and fuzzy sets

Let us consider the situation when during an experiment we can measure only upper and lower bounds of some physical quantity *x*. After the experiment we obtain a family of measurements

$$\Xi = \{A_1, ..., A_n\} \tag{1}$$

where

$$A_i = [x_i] = [x_i^-, x_i^+] = \{x : x_i^- \le x \le x_i^+\} \quad \text{for } i=1,...,n \tag{2}$$

We assume that probability of getting each measurement $[x_i]$ is the same and equal to 1/n i.e.

$$m([x_i]) = \frac{1}{n} \tag{3}$$

The pair (Ξ, m) is called a random set [4]. Ξ is called the support of the random set and *m* is a basic probability assignment. Each set $A \in \Xi$ contains the possible value of variable *x* and *m*(*A*) can be viewed as the probability that *A* is the actual range of *x*.

When measurements are intervals then theory of fuzzy sets can be applied [4,6].

Definition
Given a random set (Ξ, m), a believe function *Bel* can be defined as the following set function [4]

$$Bel(S) = \sum_{A \subset S} m(A) \tag{4}$$

Plausibility function can be defined by

$$Pl(S) = \sum_{A \cap S \neq \emptyset} m(A) \tag{5}$$

It can be shown that

$$Bel(A) = 1 - Pl(\overline{A}) \tag{6}$$

When Ξ contains only singletons then *Bel=Pl* is a probability measure (with finite support) [4].
When Ξ is a nested family i.e.

$$A_1 \subset A_2 \subset ... \subset A_n \tag{7}$$

then (Ξ, m) is called a consonant random set [4].

A fuzzy set can be defined from any random set as follows [4,6]

$$\mu(x)=\sum_{x\in A} m(A)=Pl(\{x\}) \quad (8)$$

When Ξ is nested, μ is normalised, i.e. exists such x that $\mu(x)=1$. Moreover μ is equivalent to the unique consonant random set defined by [4]

$$\Xi=\{F_1,\ldots,F_p\} \quad (9)$$

$$m(F_i)=\alpha_i-\alpha_{i+1} \quad (10)$$

where

$$F_i=\{x:\mu(x)\geq\alpha_i\} \quad (11)$$

$$\alpha_1=1>\alpha_2>\ldots>\alpha_p \quad \text{and} \quad \alpha_{p+1}=0 \quad (12)$$

3. Extension principle

A finite support random relation is a random set (Ξ_{12},m_{12}) on $X_1\times X_2$ [4]. A random relation permits to pair (x_1,x_2) of variables whose dependency can be pictured by relation $A_{12}\subset X_1\times X_2$ (where $A_{12}\in\Xi$, $(x_1,x_2)\in A_{12}$) with probability $m_{12}(A)$.

Let f be a mapping $X_1\times X_2\to X$. Now a random set (Ξ,m) can be defined in the space X in the following way:

$$A=f(A_{12}) \quad \text{where} \quad A\in\Xi, A_{12}\in\Xi_{12} \quad (13)$$

$$m(A)=m_{12}(A_{12}) \quad \text{where} \quad A=f(A_{12}) \quad (14)$$

In both spaces X and $X_1\times X_2$ we can defined fuzzy sets

$$\mu_{12}(x_1,x_2)=\sum_{(x_1,x_2)\in A_{12}} m_{12}(A_{12}) \quad (15)$$

$$\mu(x)=\sum_{x\in A} m(A) \quad (16)$$

It can be shown [4] that if the random relation (Ξ_{12},m_{12}) is consonant then the following relation holds:

$$\mu(x)=sup\{\mu_{12}(x_1,x_2): x=f(x_1,x_2)\} \quad (17)$$

$$\text{if } f^{-1}(x)=\varnothing\text{, then } \mu(x)=0 \quad (18)$$

this is so called extension principle.
A particular case of random relation is when

$$\forall A_{12}\in\Xi_{12}\ \exists A_1\in\Xi_1, A_2\in\Xi_2\ A_{12}=A_1\times A_2 \quad (19)$$

i.e. all relations are Cartesian products of subset of X_1 and X_2 respectively. Such random relation will be called 'random Cartesian products' [4].
Consonant random Cartesian products correspond to decomposable fuzzy relations. Ξ_{12} is the set of level-cuts of the equivalent fuzzy relation F_{12} on $X_1\times X_2$; these level-cuts are Cartesian products if and only if exist fuzzy sets F_1,F_2 on X_1 and X_2 such that

$$\mu_{12}(x_1,x_2)=min\{\mu_1(x_1),\mu_2(x_2)\} \quad (20)$$

where

$$\mu_1(x_1)=\sum_{x_1\in A_1} m_1(A_1) \quad (21)$$

$$\mu_2(x_1)=\sum_{x_2\in A_2} m_2(A_2) \quad (22)$$

$$m_1(A_1)=\sum_{Proj_1(A_1\times A_2)=A_1} m_{12}(A_{12}) \quad (23)$$

$$m_2(A_2)=\sum_{Proj_2(A_1\times A_2)=A_2} m_{12}(A_{12}) \quad (24)$$

where $Proj_1(A_1\times A_2)=A_1$ and $Proj_2(A_1\times A_2)=A_2$.
It can be shown that

$$\mu_1(x_1)=sup_{x_2\in X_2}\mu_{12}(x_1,x_2) \quad (25)$$

$$\mu_2(x_2)=sup_{x_1\in X_1}\mu_{12}(x_1,x_2) \quad (26)$$

For random Cartesian products extension principle can be written in the following way

$$\mu(x)=sup\{min\{\mu_1(x_1),\mu_2(x_2)\}: x=f(x_1,x_2)\} \quad (27)$$

Upper probability of same event *A* can be calculated [5]

$$Pl(A)=sup_{x\in A}\mu(x) \quad (28)$$

4. Extension principle for non-consonant random sets

If the random sets (Ξ_{12},m_{12}) aren't consonant then relation (27) is not satisfy. In this case family Ξ_{12} can be divide into parts Ξ_{12}^i. In each part condition (7) holds. For each family Ξ_{12}^i fuzzy membership function can be defined

$$\mu_{12}^i(\mathbf{x})=\sum_{\mathbf{x}\in A_{12}} m_{12}(A_{12}) \quad (29)$$

where $A_{12}\in\Xi_{12}^i$, $\mathbf{x}=(x_1,x_2)$ i=1,...,p.
Upper probability of some event $A\subset X_1\times X_2$ can be calculated using the following formula

$$Pl(\mathbf{x})=Pl(x_1,x_2)=\sum_i\mu_{12}^i(\mathbf{x})=\sum_i\mu_{12}^i(x_1,x_2) \quad (30)$$

$$Pl(A)=\sum_i sup_{(x_1,x_2)\in A}\mu_{12}^i(x_1,x_2) \quad (31)$$

or

$$Pl(A)=\sum_i sup_{(x_1,x_2)\in A}\min\{\mu_1(x_1),\mu_2(x_2)\} \quad (32)$$

Let *f* be a mapping $X_1\times X_2\to X$, then using equation (13,14) the random set (Ξ,m) in the space *X* can be calculated. Family Ξ can be divide into parts Ξ^i. In each part condition (7) hold. If we define fuzzy membership function in the following way

$$\mu^{i}(x)=\sum_{x\in A} m(A) \tag{33}$$

where $A\in \Xi^{i}$. Upper probability of some event $A\subset X$ can be calculated using the following formula

$$Pl(A)=\sum_{i} sup_{x\in A}\mu^{i}(x) \tag{34}$$

or using extension principle

$$Pl(A)=\sum_{i} sup_{\substack{x=f(x_1,x_2)\\ x\in A}} \min\{\mu_1^{i}(x_1),\mu_2^{i}(x_2)\} \tag{35}$$

Because

$$\mu^{i}(x)=\sum_{x=f(x_1,x_2)} \min\{\mu_1^{i}(x_1),\mu_2^{i}(x_2)\} \tag{36}$$

we can see that for each part Ξ^{i} (or Ξ_{12}^{i}) calculations are the same like in classical theory of fuzzy sets [11].

5. α – level-cut method for consonant random set

α-level-cat of some fuzzy number F is defined as the following interval [3]

$$F_{\alpha}=\{x:\mu_F(x)\geq\alpha\} \tag{37}$$
$$F_0=cl\{x:\mu_F(x)>0\} \tag{38}$$

where μ_F is the membership function of fuzzy set *F*. If we know α-level-cat of fuzzy number then we can calculate membership function using resolution identity [3]:

$$\mu_F(x)=sup\{\alpha:x\in F_{\alpha}\} \tag{39}$$

Finite element method leads to the following system of parameter dependent system of equation

$$\mathbf{K}(\mathbf{h})\mathbf{q}=\mathbf{Q}(\mathbf{h}) \tag{40}$$

where **K** is a stiffness matrix, **Q** is a vector of nodal forces, **q** is a vector of displacement and **h** is a vector of uncertain parameters. We assume that **h** is a vector of fuzzy parameters and calculate the following set of α-level-cat of the solution

$$\mathbf{q}_{\alpha}=\{\mathbf{q}_{\alpha}:\mathbf{K}(\mathbf{h})\mathbf{q}=\mathbf{Q}(\mathbf{h}),\mathbf{h}\in\mathbf{h}_{\boldsymbol{\alpha}}\} \tag{41}$$

To calculate the results of equation (41) sensitivity analysis method [7] or interval methods [8] can be applied.
Then we can calculate fuzzy membership function of the solution of the equation (40)

$$\mu_{q_i}(q_i)=sup\{\alpha:q_i\in q_{i\,\alpha}\} \tag{42}$$

If uncertain parameters are modelled by non-consonant random sets then we have to built a family of membership function μ^{i} using equation (33)

6. α – level-cut method for non-consonant random set

If random set $(\Xi_{\mathbf{h}},m_{\mathbf{h}})$ is non-consonant calculation are much more complicated. If we can calculate each fuzzy membership function μ^{i} separately then α-level-cat method can be applied for all these function. If we know fuzzy membership function of the displacement $\mu_{q_j}^{i}$ then upper probability that displacement q_j belongs to the interval $[q_j]=[q_j^{-},q_j^{+}]$ can be calculated using equation (34)

$$Pl([q_j])=\sum_{i} sup_{q_j\in[q_j^{-},q_j^{+}]}\mu_{q_j}^{i}(q_j) \tag{43}$$

Unfortunately usually we don't know random set $(\Xi_{\mathbf{h}},m_{\mathbf{h}})$ which describes whole vector **h**. We have only random sets (Ξ_j,m_j) which describe all coordinates h_j. Each random set (Ξ_j,m_j) we can divide into consonant parts (Ξ_j^{i},m_j^{i}) and calculate fuzzy membership function μ_j^{i}. In this circumstance we don't know what is the relation between fuzzy membership function of different coordinate h_j.
If we assume that all parameters h_j are independent then consonant random sets which describe whole vector **h** we can obtain using Cartesian products. Random set $(\Xi_{\mathbf{h}},m_{\mathbf{h}})$ we can divide into the consonant parts $\left(\Xi_{\mathbf{h}}^{i_1...i_m},m_{\mathbf{h}}^{i_1...i_m}\right)$ where

$$\Xi_{\mathbf{h}}^{i_1...i_m}=\Xi_1^{i_1}\times...\times\Xi_m^{i_m} \tag{44}$$

and

$$m_{\mathbf{h}}^{i_1...i_m}(A)=m_1^{i_1}(A_1)\cdot...\cdot m_m^{i_m}(A_m) \tag{45}$$

where $A=A_1\times...\times A_m$ and $A_k\in\Xi_k^{i_k}$. Now we can calculate fuzzy membership functions

$$\mu_{\mathbf{h}}^{i_1...i_m}(\mathbf{h})=\sum_{\mathbf{h}\in A} m_{\mathbf{h}}^{i_1...i_m}(A) \tag{46}$$

or

$$\mu_{h_j}^{i_1...i_m}(h_j)=\sum_{h_j\in A_j} m_{h_j}^{i_1...i_m}(A_j) \tag{47}$$
$$m_{h_j}^{i_1...i_m}(A_j)=\sum_{Proj_j(A)=A_j} m_{h_j}^{i_1...i_m}(A) \tag{48}$$

where $Proj_k(A)=Proj_k(A_1\times...\times A_k\times...\times A_m)=A_k$.
If we haven't got information about dependency of random sets (Ξ_j,m_j) then we should assume the worst case. In the worst case

$$m_{\mathbf{h}}^{i_1...i_m}(A)=min\{m_1^{i_1}(A_1),...,m_m^{i_m}(A_m)\} \tag{49}$$

In this case upper probability can be greater than 1. Because of this we assume that

$$Pl(A) = min\{\sum sup_{\mathbf{h}\in A} \mu_{\mathbf{h}}^{i_1 \ldots i_m}(\mathbf{h}),\ 1\} \tag{50}$$

7. Application of fuzzy sets theory to calculation of upper probability of failure of mechanical system.

The reliability of the structures can be defined in the following way [4]

$$P_f = 1 - R = P\{g(\mathbf{h}) \le 0\} \tag{51}$$

where the function g (state function) defines the state function representing the safe state and failure state [10], **h** is a vector of random parameters. In many cases only sparse and incomplete data are available. In such circumstances an interval number can be used to represent the probability measure in order to capture, in a relatively simply manner, features of fuzziness and incompleteness, so that

$$P_f \in \left[P_f^-, P_f^+\right] \tag{52}$$

where

$$P_f^+ = Pl(g \in [-\infty, 0]),\ P_f^- = Bel(g \in [-\infty, 0]) \tag{53}$$

P_f^+ can be calculated using above described algorithm. P_f^- can be calculated using Monte-Carlo simulation and the equation [4]. Another way to calculate the interval (52) is described in the paper [10].
If uncertain parameters are described by consonant random sets then upper probability of failure can be calculated using the following formula

$$P_f^+ = sup\{\mu_g(g) : g < 0\} \tag{54}$$

where μ_g is a fuzzy membership function of the state function *g*. This function can be calculated using extension principle:

$$\mu_g(g) = sup\{\mu_{h_1}(h_1) \wedge \ldots \wedge \mu_{h_m}(h_m) : g = g(h_1, \ldots, h_m)\} \tag{55}$$

If we apply α-level-cat method algorithm of calculation is the following.

1) Calculate α-level-cat of all fuzzy parameters

$$h_{i\alpha} = \{h_i : \mu_{h_i}(h_i) \ge \alpha\} \tag{56}$$

2) For all $\alpha_1, \ldots, \alpha_p$ calculate $g_\alpha = \{g(h_1, \ldots, h_m) : h_i \in h_{i\alpha}\}$. Here sensitivity analysis methods can be applied [7].
3) Using resolution identity fuzzy membership function can be calculated:

$$\mu_g(g) = sup\{\alpha : g \in g_\alpha\} \tag{57}$$

4) Using fuzzy membership function upper probability of failure can be calculated from formula (54).

8. Conclusion

In this paper a new method for modelling of uncertain parameters is presented. This method is based on the relation between fuzzy sets theory and theory of probability describes in the papers [4, 6]. The extension principle cannot be applied to fuzzy sets which was defined from non-consonant random set. This is the most important conclusion of this paper. For non-consonant random sets some special procedures (described in this paper) should be applied. If all members of family Ξ are points then operation on fuzzy sets are the same like in probability theory [9].

These results are of special interest in the framework of computational analysis. In fact, it is well known that, when incomplete statistical information is only available, it is easier to define fuzzy variables than random variables. Moreover, extended fuzzy operations are simpler than analogues operations required in the framework of probability.

Examples of application of this method will be presented on the conference.

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