Design of truss and frame structures with interval and fuzzy parameters

M.V. Rama Rao
Department of Civil Engineering
Vasavi College of Engineering
Hyderabad, Andhra Pradesh-500 031, India
dr.mvrr@gmail.com

Andrzej Pownuk
Department of Mathematical Sciences
The University of Texas at El Paso
500 West University Avenue, El Paso, Texas 79968-0514, USA
ampownuk@utep.edu

Abstract **– In this paper a method of designing a structure with interval parameters and fuzzy set parameters is presented. This paper also outlines a procedure for designing a structure with random set parameters. All procedures use solutions of the interval equations which are based on the earlier works of both authors. Safety of the structures is determined by using interval limit state equations.**

I. Design of structurs with interval parameters

According to the civil engineering codes (e.g. Eurocode [1], Load and Resistance Factor Design [2] or [11]) the structure is safe if satisfy the limit state conditions. The equations of the limit state may have the following form

$$\varphi R > \alpha D + \psi\gamma\left(\alpha_L L + \alpha_Q Q + \alpha_T T\right). \tag{1}$$

where φ is resistance factor, ψ is load Combination factor, γ is importance factor, αD is dead load factor, αL is live load factor, αQ is earthquake load factor, αT is thermal effect (temperature) load factor. In mechanical engineering some other safety such as von Mises, Drucker-Prager, Tresca or William Warke criteria can be applied [3]. In general the structure is safe if

$$g(x,p) \ge 0. \tag{2}$$

where is $g(x,p)$ some function which describes the limit state, x is special variable and p is a vector of parameters. von Misses criteria with interval parameters was discussed in the paper [4].

II. Tension-compression of structurs with interval parameters

In the case of bar under tension the structure is safe if the stress $\sigma(x)$ (where $x \in [0,L]$) in the bar is smaller by some allowable stress σ_T in tension

$$\sigma \le \sigma_T \quad \Leftrightarrow \quad \frac{N}{A} \le \sigma_t \tag{3}$$

where N is an axial force, x is space coordinate and A is an area of cross-section. If the structure contains interval parameters (e.g. area of cross-section, Young modulus, forces, geometrical dimensions) $p = (p_1, ..., p_m)$ then the structure is safe if

$$\sigma(x,p) \le \sigma_t \tag{4}$$

for all $p \in \left[\underline{p}, \overline{p}\right], x \in \Omega$.

The condition (4) is satisfied if

$$\forall x \in \Omega, \overline{\sigma}(x) \le \sigma_t. \tag{5}$$

where

$$\underline{\sigma}(x) = \inf\left\{\sigma(x,p) : p \in \left[\underline{p}, \overline{p}\right]\right\}, \tag{6}$$

$$\overline{\sigma}(x) = \sup\left\{\sigma(x,p) : p \in \left[\underline{p}, \overline{p}\right]\right\}. \tag{7}$$

If we take into account tension and compression then the safety condition has the form

$$\forall x \in \Omega, \forall p \in \left[\underline{p}, \overline{p}\right],\ \sigma(x) \le \sigma_t. \tag{8}$$

in tension and

$$\forall x \in \Omega, \forall p \in \left[\underline{p}, \overline{p}\right],\ \left|\sigma(x)\right| \le \sigma_c. \tag{9}$$

in compression. Then in order to check the safety of the structure with the interval parameters we need to know the value of the interval stress $\sigma(x) \in \left[\underline{\sigma}(x), \overline{\sigma}(x)\right]$.

III. Example – Bar with interval parameters

Let us consider a bar under tension which is shown in the Fig. 1

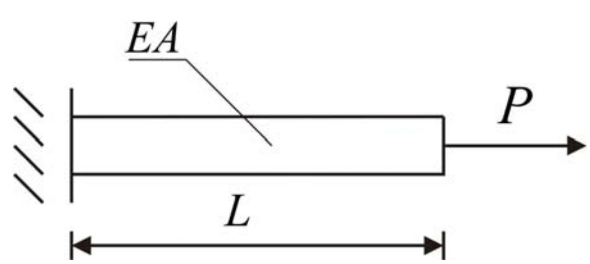

Fig. 1- Bar under tension

The $E \in [210, 212]\,GPa$, $A \in [0.0024, 0.0026]\,m^2$, $\sigma_T \in [250, 252]\,MPa$. $P \in [550, 590]\,kN$. The structure is safe if

$$\frac{\overline{P}}{\underline{A}} = \overline{\sigma} \le \underline{\sigma}_T \tag{10}$$

In this case $\overline{\sigma} = 245.8\,MPa$, $\underline{\sigma}_T = 250\,MPa$ then the structure is safe.

Example – Bar with random set parameters (upper probability approach)

Now let's consider the same structure but now the force P is described by the following random set $P(\omega_1) = [550, 560]kN$, $P(\omega_2) = [550, 570]kN$, $P(\omega_3) = [560, 590]kN$, $P(\omega_4) = [520, 570]kN$. The area of cross-section is described by the following random set $A(\gamma_1) \in [2.4, 2.7]10^{-3}m^2$, $A(\gamma_2) \in [2.5, 2.7]10^{-3}m^2$, $A(\gamma_3) \in [2.2, 2.6]10^{-3}m^2$. We assume that $P_\Omega\{\omega_1\} = P_\Omega\{\omega_2\} = P_\Omega\{\omega_3\} = P_\Omega\{\omega_4\} = \frac{1}{4}$ and $P_\Gamma\{\gamma_1\} = P_\Gamma\{\gamma_2\} = P_\Gamma\{\gamma_3\} = \frac{1}{3}$. Upper probability of failure can be calculated form the following condition [6]

$$\overline{P}_f = P_{\Omega\times\Gamma}\left\{(\omega,\gamma) : \frac{\overline{P}(\omega)}{\underline{A}(\gamma)} > \underline{\sigma}_T\right\} = \frac{4}{12} = 0.33 \tag{11}$$

Table 1 - Interval stress

$\sigma(\omega,\gamma)$	Value $[MPa]$
$\sigma(\omega_1,\gamma_1)$	[203.704, 233.333]
$\sigma(\omega_2,\gamma_1)$	[203.704, 237.500]
$\sigma(\omega_3,\gamma_1)$	[207.407, 245.833]
$\sigma(\omega_4,\gamma_1)$	[192.593, 237.500]
$\sigma(\omega_1,\gamma_2)$	[203.704, 224.000]
$\sigma(\omega_2,\gamma_2)$	[203.704, 228.000]
$\sigma(\omega_3,\gamma_2)$	[207.407, 236.000]
$\sigma(\omega_4,\gamma_2)$	[192.593, 228.000]
$\sigma(\omega_1,\gamma_3)$	[211.538, 254.545]
$\sigma(\omega_2,\gamma_3)$	[211.538, 259.091]
$\sigma(\omega_3,\gamma_3)$	[215.385, 268.182]
$\sigma(\omega_4,\gamma_3)$	[200.000, 259.091]

IV. Example – Bar with random set parameters - (Clouds theory approach)

Let us consider the same data as in the previous example and calculate the following upper probability.

$$F(\sigma) = P_{\Omega\times\Gamma}\left\{(\omega,\gamma) : \frac{\overline{P}(\omega)}{\underline{A}(\gamma)} > \sigma\right\} \tag{12}$$

The graph of the function $F(\sigma)$ is on the Fig. 2.

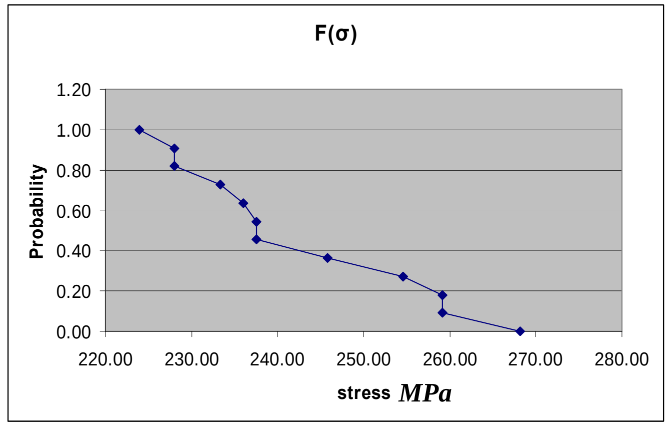

Fig. 2- Graph of the function $F(\sigma)$

Now upper probability of failure can be calculated as

$$\overline{P}_f = F(\underline{\sigma}_T) = \frac{4}{12} \tag{13}$$

Above results are consistent with the theory of clouds which was introduced by the Arnold Neumaier [7].

V. Design of truss structures

In the case of truss structures the condition (8) or (9) must be satisfied in all bars. Let us consider a truss structure which is shown in the Fig. 3. Number of bars are shown on the Fig. 4.

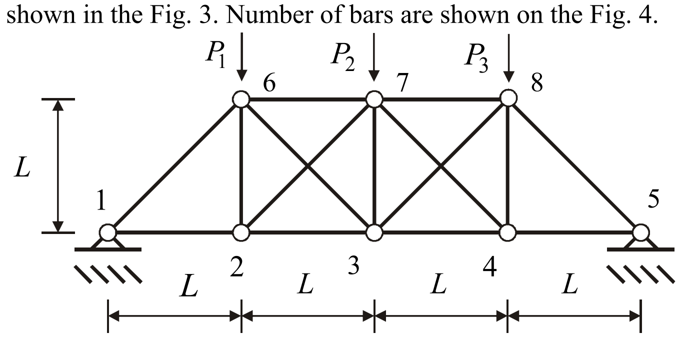

Fig. 3- 15 element truss structure

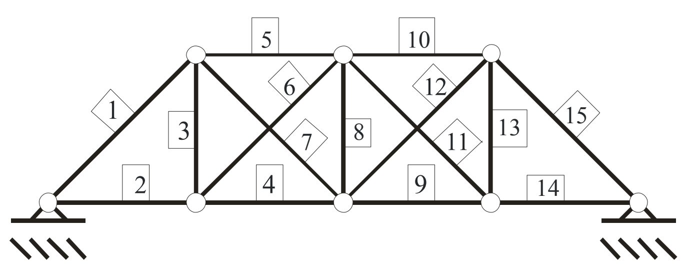

Fig. 4- Elements of truss structure

The data in the calculations are the following: $E \in [199, 201] GPa$, $A \in 0.06\, m^2$, $P \in [96,104] kN$. The interval axial forces are given in the Table 2. In calculations we assume that $P = \gamma P_0$ where

TABLE 2 - INTERVAL STRESS
(COMBINATORIAL APPROACH, DEPENDENT PARAMETERS)

No. of element	$\underline{\sigma}_i$ $[MPa]$	$\bar{\sigma}_i$ $[MPa]$
1	-3.680	-3.390
2	-0.377	-0.348
3	0.695	0.753
4	0.348	0.377
5	-2.710	-2.500
6	-1.070	-0.983
7	0.148	0.160
8	-0.227	-0.209
9	0.348	0.377
10	-2.710	-2.500
11	-1.070	-0.983
12	0.148	0.160
13	0.695	0.753
14	-0.377	-0.348
15	-3.680	-3.390

All stresses in the Table 2 are smaller than allowable stress $\sigma_0 = 250 MPa$ then the structure is safe.

In above described example the Young modulus and forces are all dependent. If we consider the case of independent forces and Young modulus we have the results which are shown in the Table 3.

TABLE 3 - INTERVAL STRESS IN TRUSS
(COMBINATORIAL APPROACH, INDEPENDENT CASE)

No. of element	$\underline{\sigma}_i$ $[MPa]$	$\bar{\sigma}_i$ $[MPa]$
1	-3.680	-3.390
2	-0.410	-0.314
3	0.684	0.765
4	0.327	0.397
5	-2.720	-2.500
6	-1.080	-0.967
7	0.107	0.201
8	-0.259	-0.177
9	0.327	0.397
10	-2.720	-2.500
11	-1.080	-0.967
12	0.107	0.201
13	0.684	0.765
14	-0.410	-0.314
15	-3.680	-3.390

Because stress in all bars is smaller than allowable stress then the structure is safe. The results are shown in the Tables 2 and 3 was calculated using combinatorial approach [5]. The results which calculated using sensitivity analysis in the case of independent parameters are shown in the Table 4.

TABLE 4 - INTERVAL STRESS IN TRUSS
(INDEPENDENT CASE, SENSITIVITY ANALYSIS)

No. of element	$\underline{\sigma}_i$ $[MPa]$	$\bar{\sigma}_i$ $[MPa]$
1	-3.680	-3.390
2	-0.409	-0.317
3	0.687	0.764
4	0.330	0.392
5	-2.720	-2.500
6	-1.080	-0.971
7	0.109	0.201
8	-0.254	-0.178
9	0.332	0.396
10	-2.720	-2.500
11	-1.080	-0.968
12	1.120	0.196
13	0.685	0.764
14	-0.409	-0.317
15	-3.680	-3.390

The results are good approximation of the combinatorial solution and they indicate that the structure is safe.

VI. DESIGN OF FRAME STRUCTURES WITH INTERVAL PARAMETERS

In the case of frame structures the maximum stress can be calculated as a combination of bending and tension

$$\sigma = \frac{N}{A} \pm \frac{M}{Z} \tag{14}$$

where M is bending moment, N is axial force, A is area of cross-section and Z is section modulus. In the simplest case Z can be calculated as

$$Z = \frac{J}{z_{\max}} \tag{15}$$

The structure is safe if

$$\frac{N}{A} \pm \frac{M}{Z} \le \sigma_{yt} \tag{16}$$

in tension and

$$\left| \frac{N}{A} \pm \frac{M}{Z} \right| \le \sigma_{yc} \tag{17}$$

in compression, where σ_{yc} is a yield stress in compression and σ_{yt} is a yield stress in tension.

VII. Frame structure example

A single bay single storey portal frame as shown in Fig. 5 is chosen as an illustrative example.

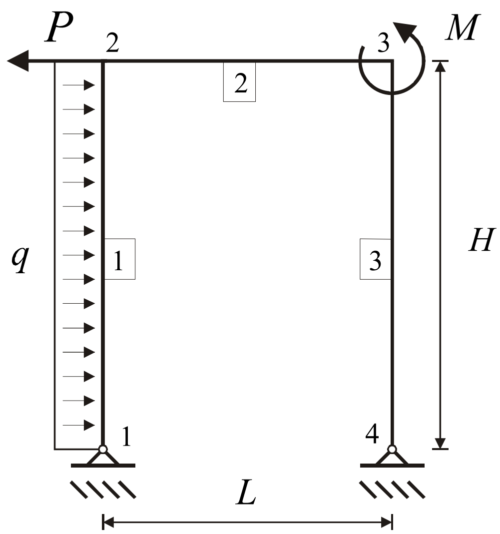

Fig. 5- Portal frame

In the case of interval uncertainty two different approaches are presented.

In first approach the interval bending moment and the interval axial force will be computed and then the condition (16) or (17) will be checked. The interval internal forces can be calculated using the method introduced by Rama Rao [8], combinatorial approach or sensitivity analysis [5].

In the second approach the condition is checked directly. In this case the left side of the equation (16) or (17) is calculated combinatorial approach or sensitivity analysis [5] and then the whole interval relations (16) or (17) are checked.

In calculations the following numerical data was used: $\sigma_y = 250MPa$, $E_1 \in [199, 201]GPa$, $E_2 \in [199, 201]GPa$, $E_3 \in [199, 201]GPa$, $A_1 = 0.01m^2$, $A_2 = 0.01m^2$, $A_1 = 0.01m^2$, $J_1 = 8.333 \cdot 10^{-6} m^4$, $J_2 = 8.333 \cdot 10^{-6} m^4$, $J_3 = 8.333 \cdot 10^{-6} m^4$, $z_{\max} = 0.05m$, $q \in [2.4, 2.6] kN/m^2$, $P \in [9.6, 10.4] kN$, $M \in [4.9, 5.1] kNm$. Cross-section is a square ($0.1 \times 0.1m^2$).

Interval bending moment and axial force are shown in the Table. 5. The values were calculated in the nodal points.

Table 5 - Internal forces

$\underline{N}$ $[kN]$	$\overline{N}$ $[kN]$	$\underline{M}$ $[kNm]$	$\overline{M}$ $[kNm]$
-5.08	-5.78	3.80	2.08
-5.78	-5.08	-8.30	-7.33
2.62	2.01	-7.33	-8.30
2.01	2.62	7.90	9.06
5.78	5.08	4.03	2.93
5.08	5.78	-6.45	-5.10

Table 6 - Maximum stresses (indirect approach)

$\frac{N}{A} - \frac{M}{Z}$ $[MPa]$		$\frac{N}{A} + \frac{M}{Z}$ $[MPa]$	
-23.329	-12.985	11.898	22.243
49.208	43.402	-50.294	-44.488
44.242	50.048	-43.780	-49.585
-54.145	-47.211	47.673	54.608
-23.594	-17.006	18.092	24.680
31.082	39.212	-38.126	-29.996

The results from the endpoint combination method are in the Table 7.

Table 7 - Maximum stresses (combinatorial approach)

$\frac{N}{A} - \frac{M}{Z}$ $[MPa]$		$\frac{N}{A} + \frac{M}{Z}$ $[MPa]$	
-23.399	-12.986	11.968	22.245
43.498	49.173	-50.330	-44.514
44.210	50.010	-49.493	-43.802
-54.075	-47.217	47.625	54.592
-23.601	-17.071	18.099	24.745
31.090	39.277	-38.134	-30.061

VIII. Rama Rao's method

In the case of frame structures the maximum stress can be calculated as a combination of bending and tension

$$\sigma = \sigma_1 \pm \sigma_2 \tag{18}$$

where direct stress $\sigma_1 = \frac{N}{A}$ and bending stress $\sigma_2 = \frac{M}{Z}$,where M is bending moment, N is axial force, A is area of cross-section and Z is section modulus. In the simplest case Z can be calculated as

$$Z = \frac{J}{z_{\max}} \tag{19}$$

As per the Indian steel code IS 800-1984 [11], the structure is safe if

$$\sigma_1 \le \sigma_{t,allow} \tag{20}$$

in tension and

$$\sigma_1 \le \sigma_{c,allow} \tag{21}$$

in compression and

$$\sigma_2 \le \sigma_{b,allow} \tag{22}$$

in bending, subject to the condition

$$\frac{\sigma_1}{\sigma_{t,allow}} + \frac{\sigma_2}{\sigma_{b,allow}} \le 1 \tag{23}$$

for combined axial tension and bending and

$$\frac{\sigma_1}{\sigma_{c,allow}} + \frac{\sigma_2}{\sigma_{b,allow}} \leq 1 \qquad (24)$$

for combined axial compression and bending, where $\sigma_{t,allow} = 0.6 f_y$ and $\sigma_{b,allow} = 0.66 f_y$, $\sigma_{c,allow} = \min\,(0.6 f_y, \sigma_{ac})$ where

$$\sigma_{ac} = 0.6 f_y \frac{f_{cc} f_y}{\left[f_{cc}^n + f_y^n\right]^{\frac{1}{n}}} \quad , f_{cc} = \frac{\Pi^2 E}{\lambda^2} \qquad (25)$$

where n = 1.4 , E is the Young's modulus, λ is the slenderness ratio of the member and f_y is the Yield stress.

IX. Example problem - 2

A single bay- single storey portal frame as shown in figure below is chosen as an illustrative example. Columns have a cross section of 0.3m×0.2m and beam has a cross section of 0.3m×0.25m. Young's modulus is 200GPa. Load P = 150kN and q = 37.5kN/m.

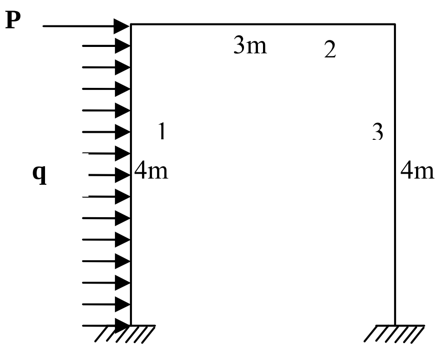

Fig. 6- Portal frame

The membership function for load and Young's modulus are shown in figures 7 and 8.

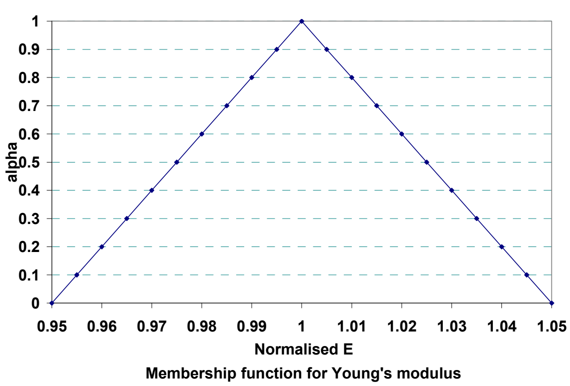

Fig. 7

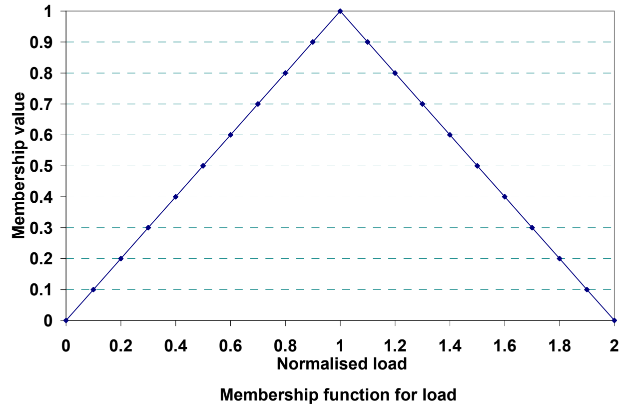

Fig. 8

The structural response of the portal frame is obtained at various levels of membership value ranging from 0 through 1 using the procedure developed by Rama Rao [8]. Interval values of axial force and bending moment are computed using this procedure. Table 8 shows the normalised interval values of load and Young's modulus adopted at each level of α.
For example, at $\alpha = 0.4$, loads are $P = 150[0.4,1.6]$ *kN*, $q = 37.5[0.4,1.6]$ *kN/m* and $E = 200[0.97,1.03]$ *GPa*. The corresponding values of axial force and bending moment in element 1 at node 2 are shown in Table 9.
Combined membership functions are plotted for axial stress and bending stress and combined stress using the procedure described by Moens and Vandepitte [9] and earlier by Buckley [10].
These stresses are plotted in Fig. 9, 10 and 11. Fig 9 and 10 show the combined membership functions for axial stress and bending stress. In figure 10, the thick vertical line shows the allowable bending stress. Fig. 11 shows a plot of combined stresses σ_1 and σ_2. In this figure, the thick horizontal line at α =0.2 represents the limit interval stresses satisfying the above equation.

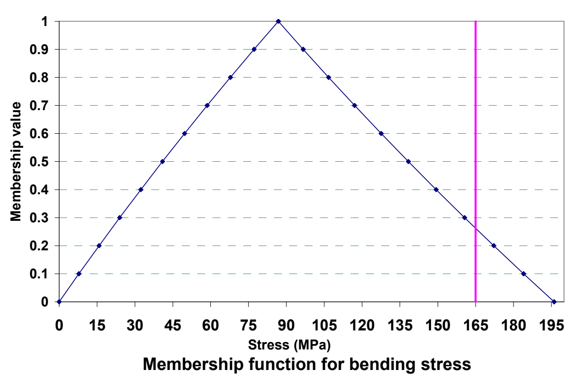

Fig. 9

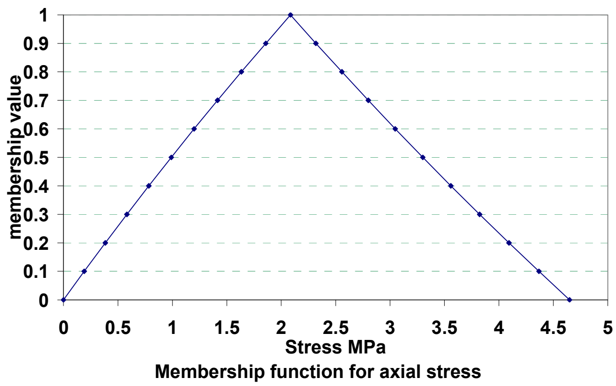

Fig.. 10

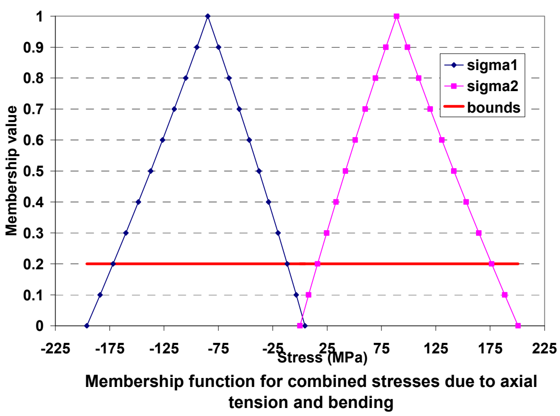

Fig.. 11

TABLE 8 - UNCERTAIN INPUT DATA

α	Load Normalised	E Normalised
0	[0.0,2.0]	[0.950,1.050]
0.1	[0.1,1.9]	[0.955,1.045]
0.2	[0.2,1.8]	[0.960,1.040]
0.3	[0.3,1.7]	[0.965,1.035]
0.4	[0.4,1.6]	[0.970,1.030]
0.5	[0.5,1.5]	[0.975,1.025]
0.6	[0.6,1.4]	[0.980,1.020]
0.7	[0.7,1.3]	[0.985,1.015]
0.8	[0.8,1.2]	[0.990,1.010]
0.9	[0.9,1.1]	[0.995,1.005]
1	[1.0,1.0]	[1.000,1.000]

TABLE 9 - INTERVAL FORCES AND MOMENTS IN ELEMENT 1

α	Axial Force (*kN*)	Bending Moment (*kNm*)
0	[0,185.9]	[0, 26.45]
0.1	[7.5,174.7]	[10.4, 245.3]
0.2	[15.3,163.7]	[21.0, 229.5]
0.3	[23.2,152.9]	[31.9, 214.1]
0.4	[31.3,142.3]	[43.1, 199.1]
0.5	[39.5,132.0]	[54.5, 184.4]
0.6	[47.9,121.9]	[66.2, 170.0]
0.7	[56.5,111.9]	[78.2, 156.0]
0.8	[65.3,102.2]	[90.4, 142.3]
0.9	[74.3,92.7]	[102.9, 128.8]
1	[83.4,83.4]	[115.7, 115.7]

X. CONCLUSIONS

Numerical results obtained in this paper indicate that it is possible to design frame and truss structures with the interval, fuzzy and random sets parameters. In order to get the value of the uncertain limit state, combinatorial method, sensitivity analysis and indirect method were applied. Numerical results reveal small differences between the indirect method which is based on the interval arithmetic and the combinatorial method. This difference is caused by the overestimation of the interval evaluation of the limit state equation, in presented examples that difference is very small. Fuzzy limit state is treated as family of interval limit states. Presented approach can be extended to the design of 2D or 3D structures with a given limit state equations.

REFERENCES

[1] A. W. Beeby, R. S. Narayanan, Designers' Handbook to Eurocode 2, Thomas Telford, 1995
[2] Samuel G. Paikowsky, Load and Resistance Factor Design (LRFD) for Deep Foundations ,Transportation Research Board, 2004
[3] J. R. Barber, Elasticity, Springer 2002
[4] Gang Xiang, Andrzej Pownuk, Scott A. Starks, Olga Kosheleva Von Mises Failure Criterion for Ductile Materials: How to Efficiently Use it under Interval and Fuzzy Uncertainty Proceedings of the Conference NAFIPS'07 , San Diego, California, USA, June 24-27, 2007, pp. 570-575
[5] Pownuk A., Numerical solutions of fuzzy partial differential equation and its application in computational mechanics, Fuzzy Partial Differential Equations and Relational Equations: Reservoir Characterization and Modeling (M. Nikravesh, L. Zadeh and V. Korotkikh, eds.), Studies in Fuzziness and Soft Computing, Physica-Verlag, 2004, pp. 308-347
[6] Dempster, A.P., 1967, Upper and Lower Probabilities Induced by a Multivalued Mapping, Annals of Math. Stat. 38:325-339, 1967
[7] Neumaier A., Clouds, fuzzy sets and probability intervals, Reliable Computing 10: 249–272, 2004
[8] M.V. Rama Rao and R. Ramesh Reddy, Fuzzy finite element analysis of structures with uncertainty in load and material properties, Journal of Structural Engineering, June-July 2006, Vol. 33, No.2, pp. 129-137
[9] David Moens, Dirk Vandepitte, A fuzzy finite element procedure for the calculation of uncertain frequency-response functions of damped structures: Part 1—Procedure, Journal of Sound and Vibration 288 (2005) 431–462
[10]Buckley J.J., Qu Y., On using α-cuts to evaluate fuzzy equations. Fuzzy Sets and Systems, Vol.38, 1990, pp.309-312
[11]IS 800-1984 Indian Standard code of Practice for general steel construction, Bureau of Indian Standards, New Delhi, India