Finite Difference Equations with the Interval Parameters

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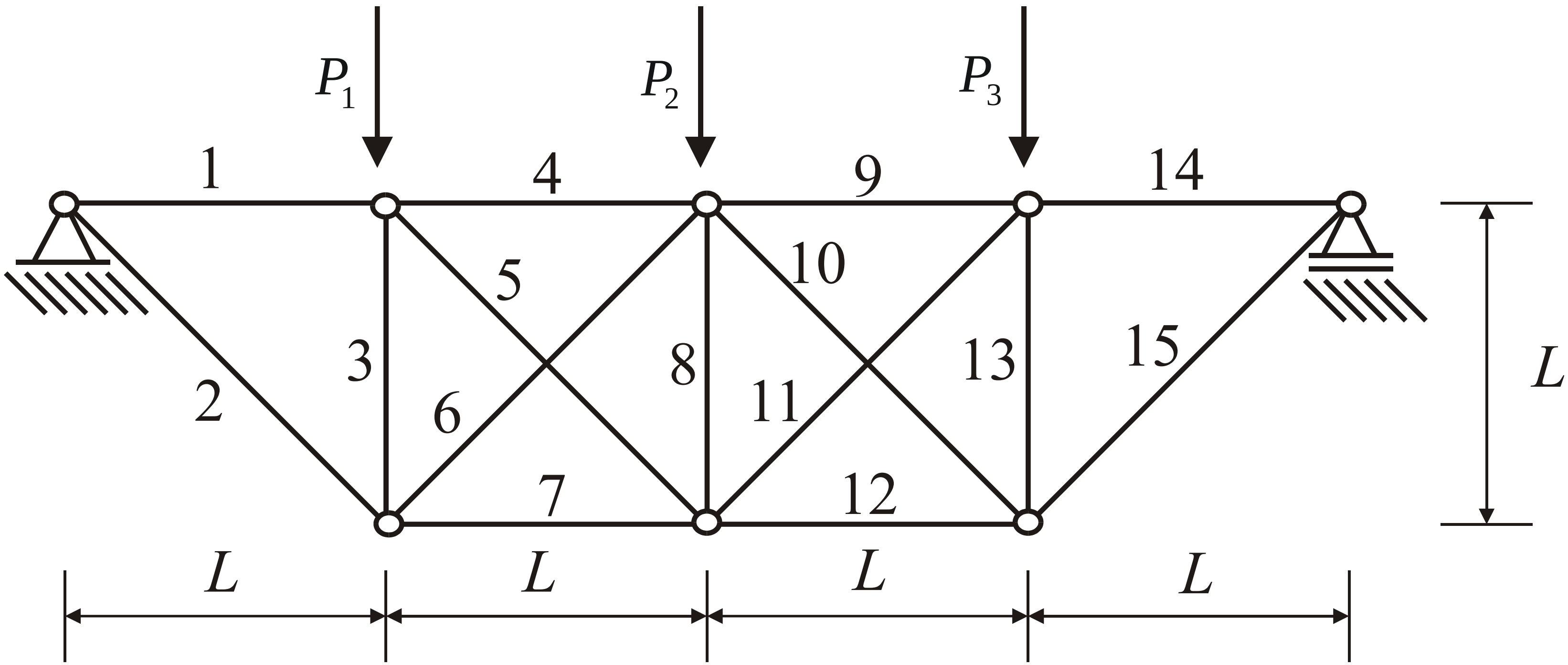
Outline

- Modeling of uncertainty
- Set valued solutions
- Automated theorem proving

Truss structure

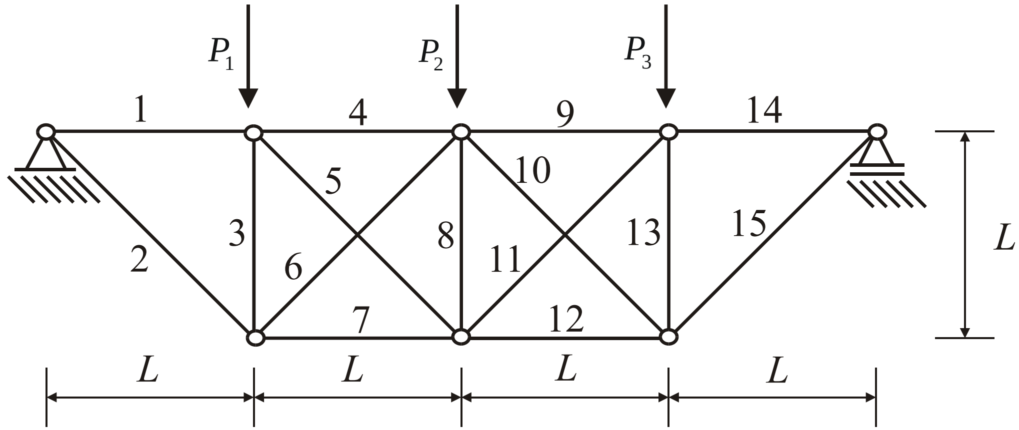


Perturbated forces

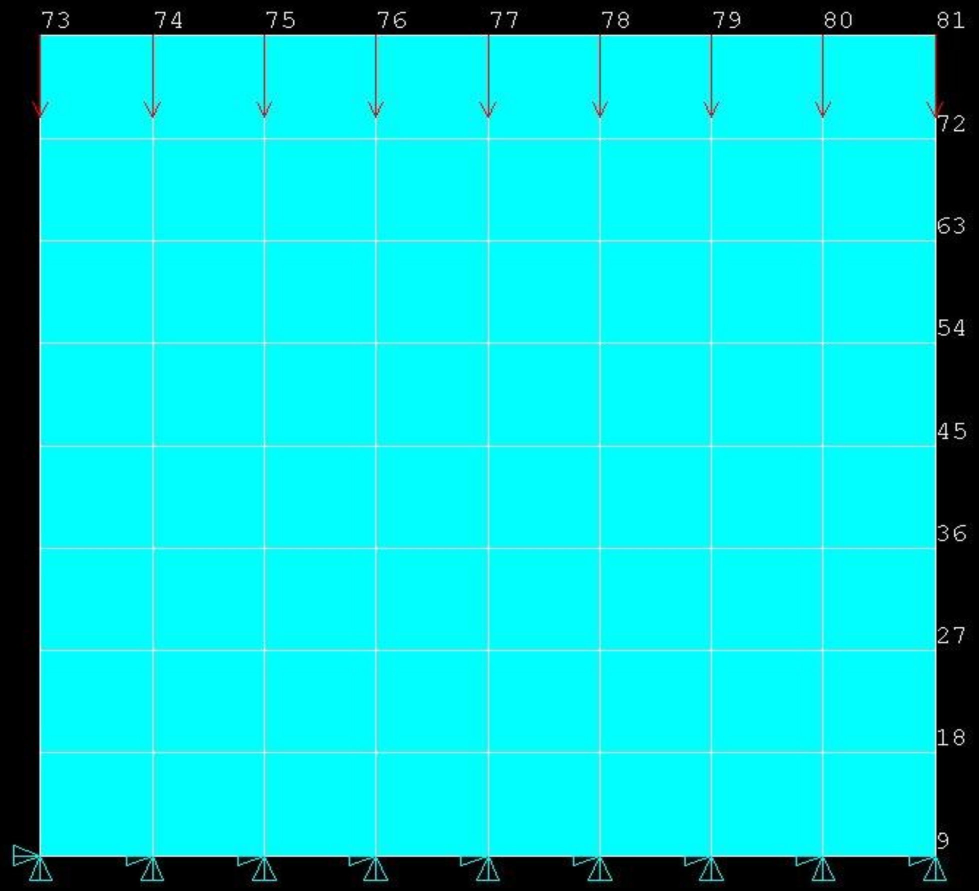
$$P = P_0 \pm \Delta P$$

5% uncertainty

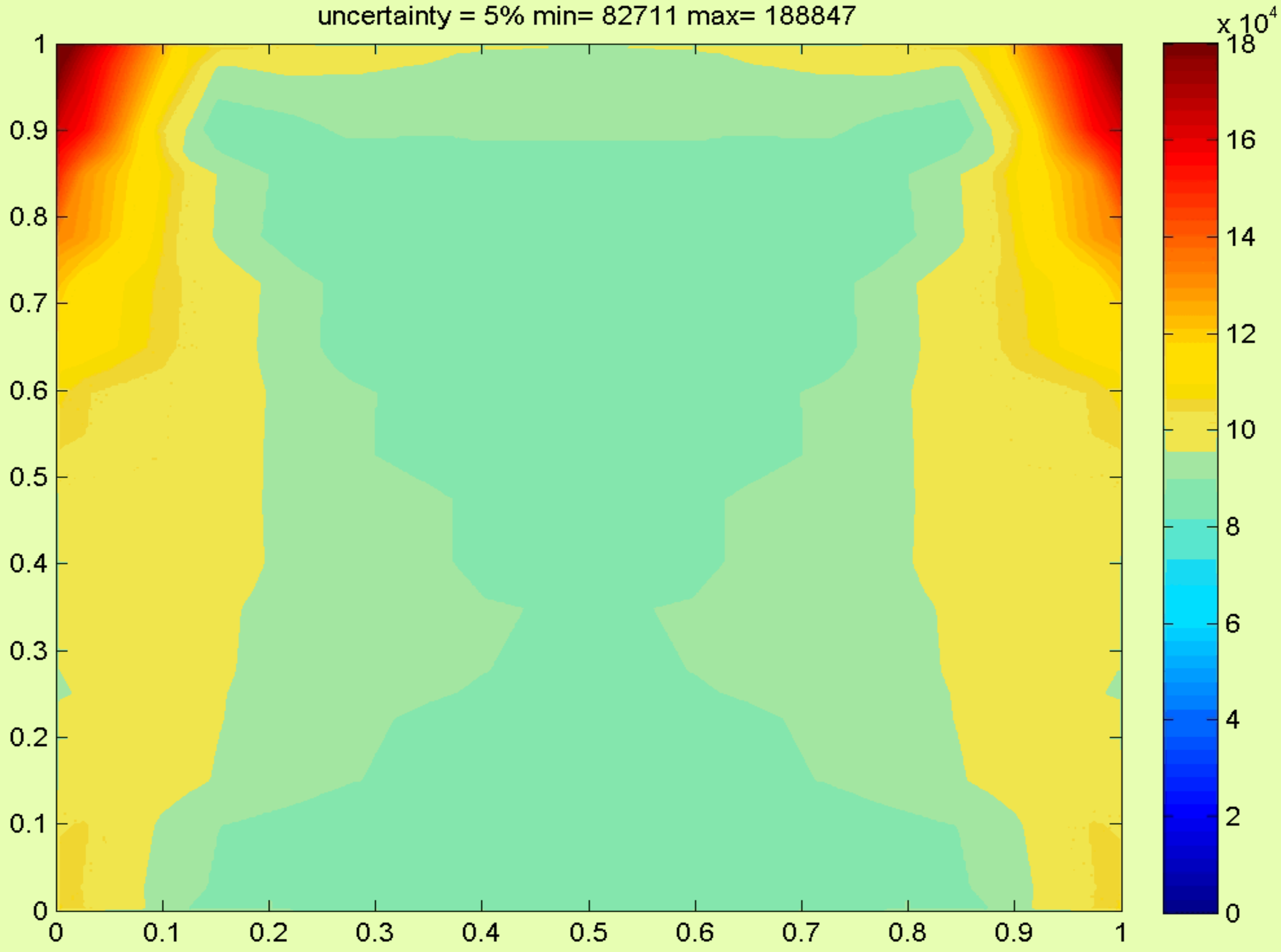
No	1	2	3	4	5	6	7	8
ERROR %	10	9,998586	10,00184	10,00126	60,18381	11,67825	9,998955	31,8762
No	9	10	11	12	13	14	15	
ERROR %	10,00126	11,67825	60,18381	9,998955	10,00184	10	9,998586	



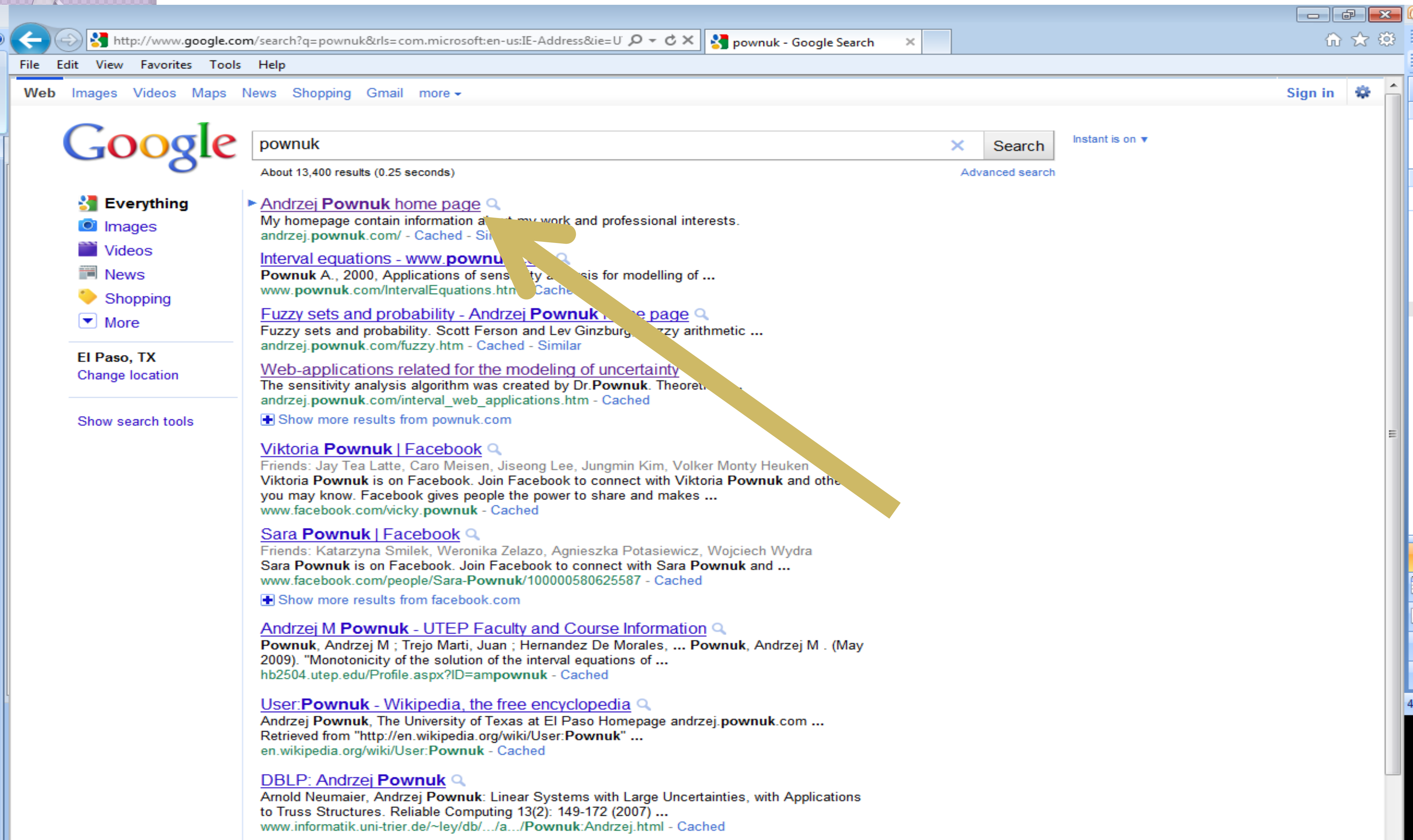
Example II (model)



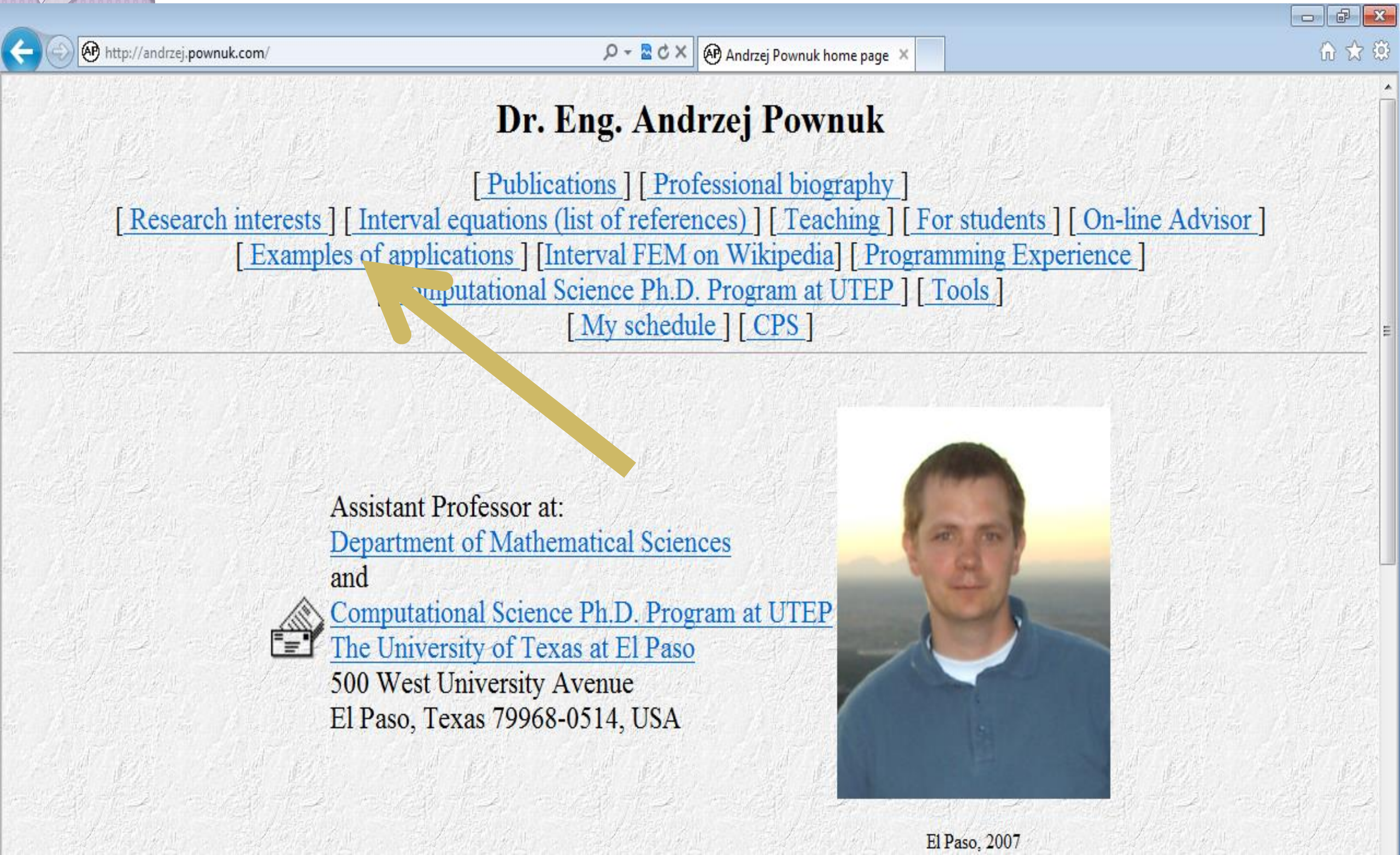
Example II (results)



More examples



More examples



http://calculus.math.utep.edu/AdaptiveTaylorSeries-1.1/default.aspx

utep.edu

Second order approximaiton

u''+(k/m)*u=(P/m)*cos(w*t)

v=u'

Calculation

u0= 1 v0= 1

P= 1

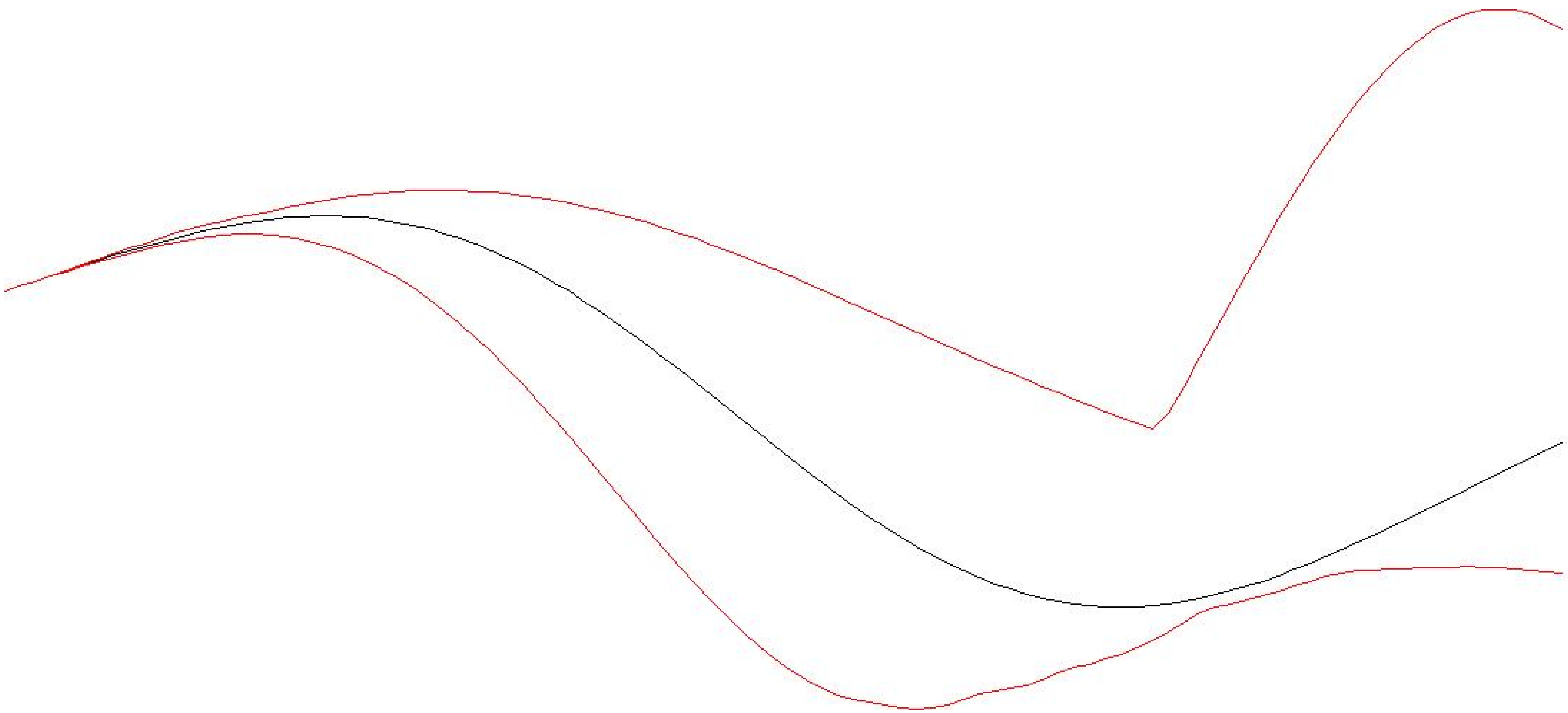
kMin= 1 kMax= 2

mMin= 1 mMax= 2

w= 2

number of intermediate points = 5

dt = 0.05 number of timesteps = 100



http://andrzej.pownuk.com/silverlight/VibrationsWithIntervalParameters/VibrationsW

VibrationsWithIntervalPara...

E= 200E9 dE= 5 % min E = 190000000000 max E = 210000000000 rho0 = 7874 [kg/m^3]

A= 0.01 dA = 5 % min A = 0.0095 max A = 0.0105 drho = 5 %

J = 8.333E-6 dJ = 5 % min J = 7.91635E-06 max J = 8.74965E-06 min rho= 7480.3

dt= 0.001 [s] P = 1000 [N] Time steps for load = 1 max rho= 8267.7

L = 10.0 [m] n= 1 Total time when the load was applied = 0.001 [s]

Ln= 5 [m]

Init calculations

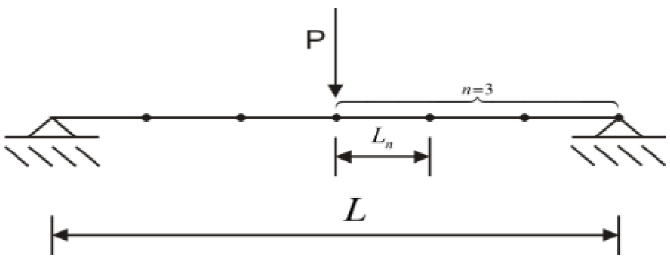
Number of interval parameters: 8

List of nodes

Number of timesteps = 600

node 1, x = 0
node 2, x = 5
node 3, x = 10

Number of DOF = 9



number of elements = 2

Number of nodes = 3

Dof in nodes

node 0
0 1 2
node 1
3 4 5
node 2
6 7 8

Nodes in elements

element 0
0 1
element 1
1 2

DOF in elements

element 0
0 1 2
3 4 5
element 1
3 4 5
6 7 8

Mid point solution

Newmark method 's parameters: beta = 0.5 gamma = 0.5

Solution to calculate, node = 1 DOF = 1

Stifness matrix

Mass matrix

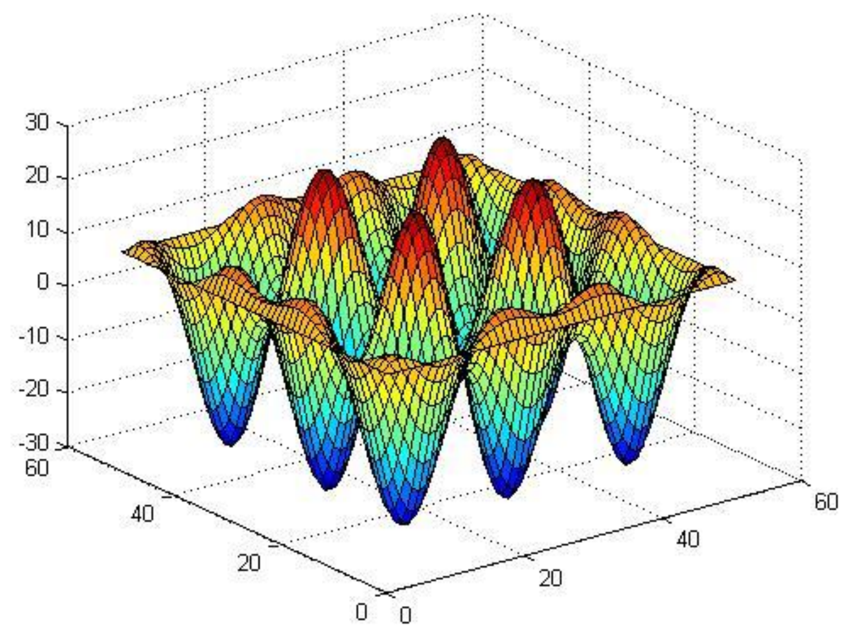
Show local stifness matrix 0

Show local mass matrix 0

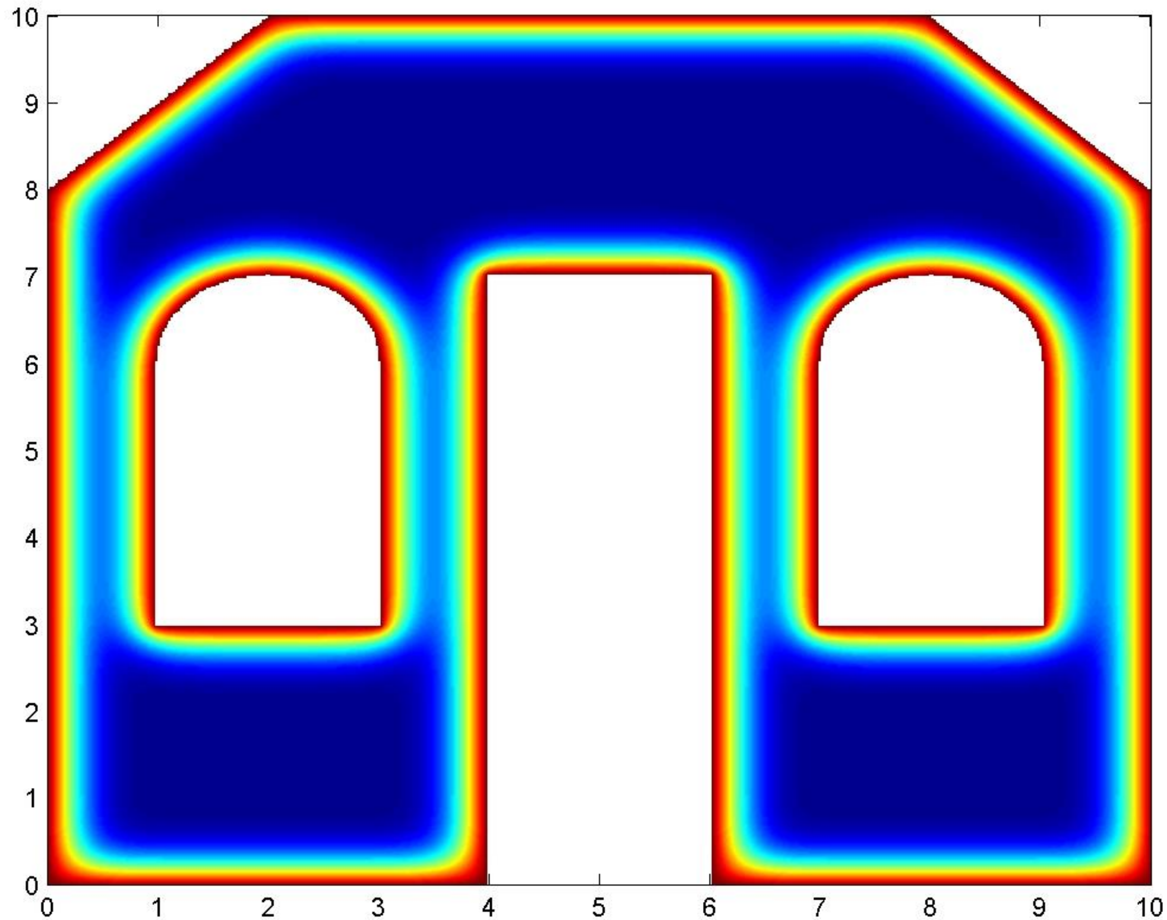
2D PDE

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + T = f$$

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} + T_{i,j} = f_{i,j}$$



Solution of 2D heat equation

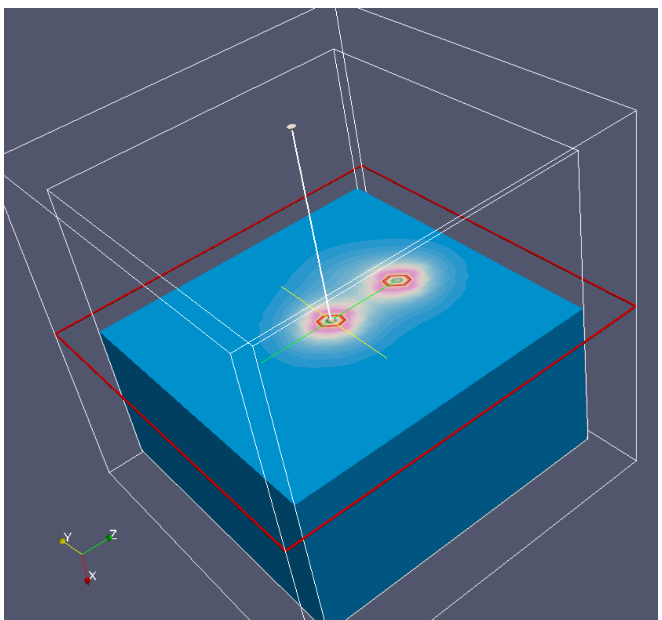


3D Heat transfer problem

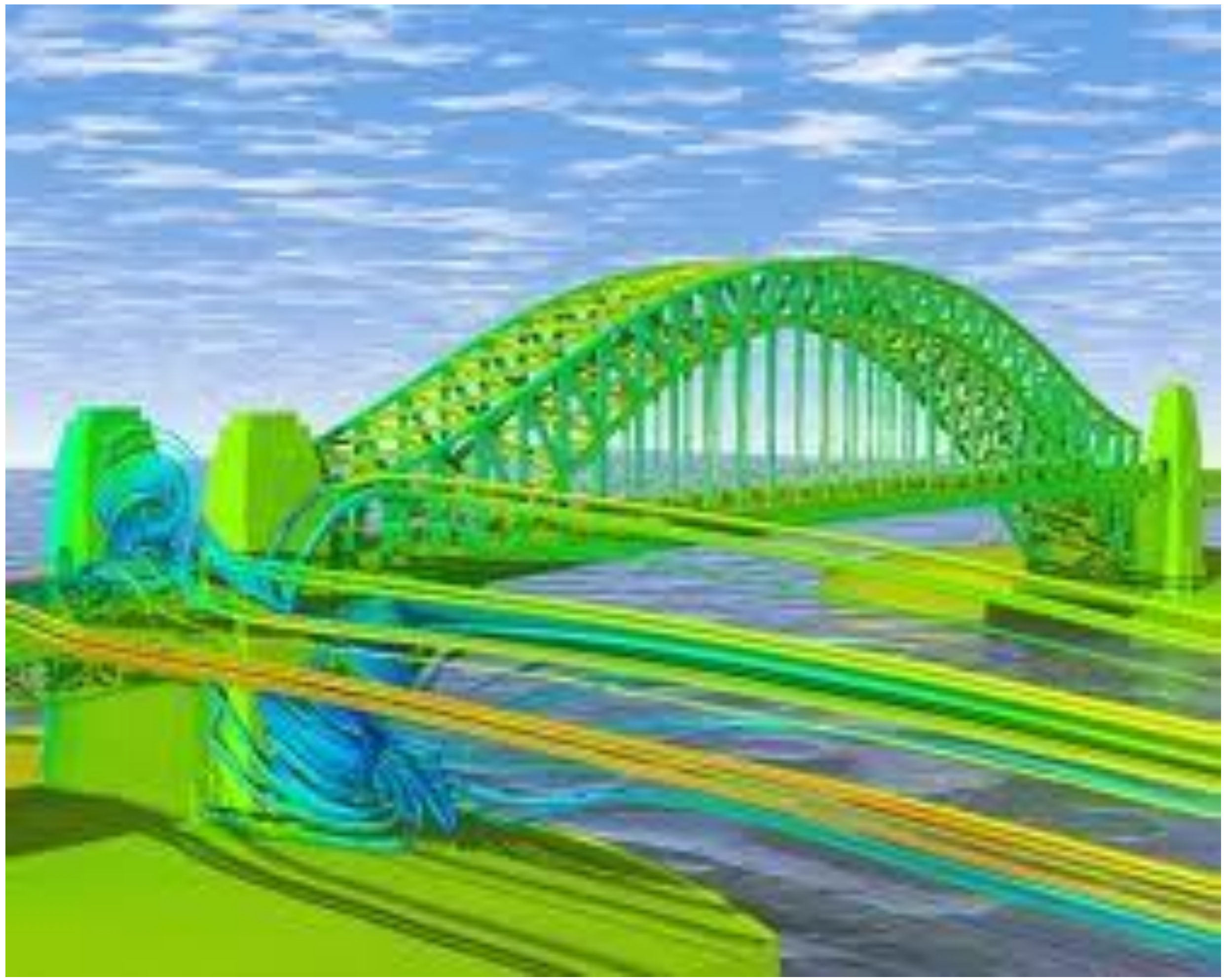
$$\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+q=\frac{\partial T}{\partial t}$$

$$\alpha\left(\frac{T_{i+1,j,k}{}^t-2T_{i,j,k}{}^t+T_{i-1,j,k}^t}{\Delta x^2}+\frac{T_{i,j+1,k}{}^t-2T_{i,j,k}{}^t+T_{i,j-1,k}^t}{\Delta y^2}+\frac{T_{i,j,k+1}{}^t-2T_{i,j,k}{}^t+T_{i,j,k-1}^t}{\Delta z^2}\right)+q_{i,j,k}=\frac{T_{i,j,k}^{t+\Delta t}-T_{i,j,k}^t}{\Delta t}$$

$$T_{i,j,k}^{t+\Delta t}=T_{i,j,k}^t+\alpha\left(\frac{T_{i+1,j,k}{}^t-2T_{i,j,k}{}^t+T_{i-1,j,k}^t}{\Delta x^2}+\frac{T_{i,j+1,k}{}^t-2T_{i,j,k}{}^t+T_{i,j-1,k}^t}{\Delta y^2}+\frac{T_{i,j,k+1}{}^t-2T_{i,j,k}{}^t+T_{i,j,k-1}^t}{\Delta z^2}\right)\Delta t+q_{i,j,k}\Delta t$$



Engineering simulations



Uncertain parameters:

$$ax = b \qquad x = \frac{b}{a}$$

Example: $[1,2]x = [1,4]$

$$x = \frac{[1,4]}{[1,2]} = [1,2]$$

Extreme values of montone functions

$$f(x) = x^2, \quad x \in [1, 2]$$

$$\frac{df(x)}{dx} = 2x \in [2, 4]$$

$$\underline{f} = f(\underline{x}) = f(1) = 1^2 = 1$$

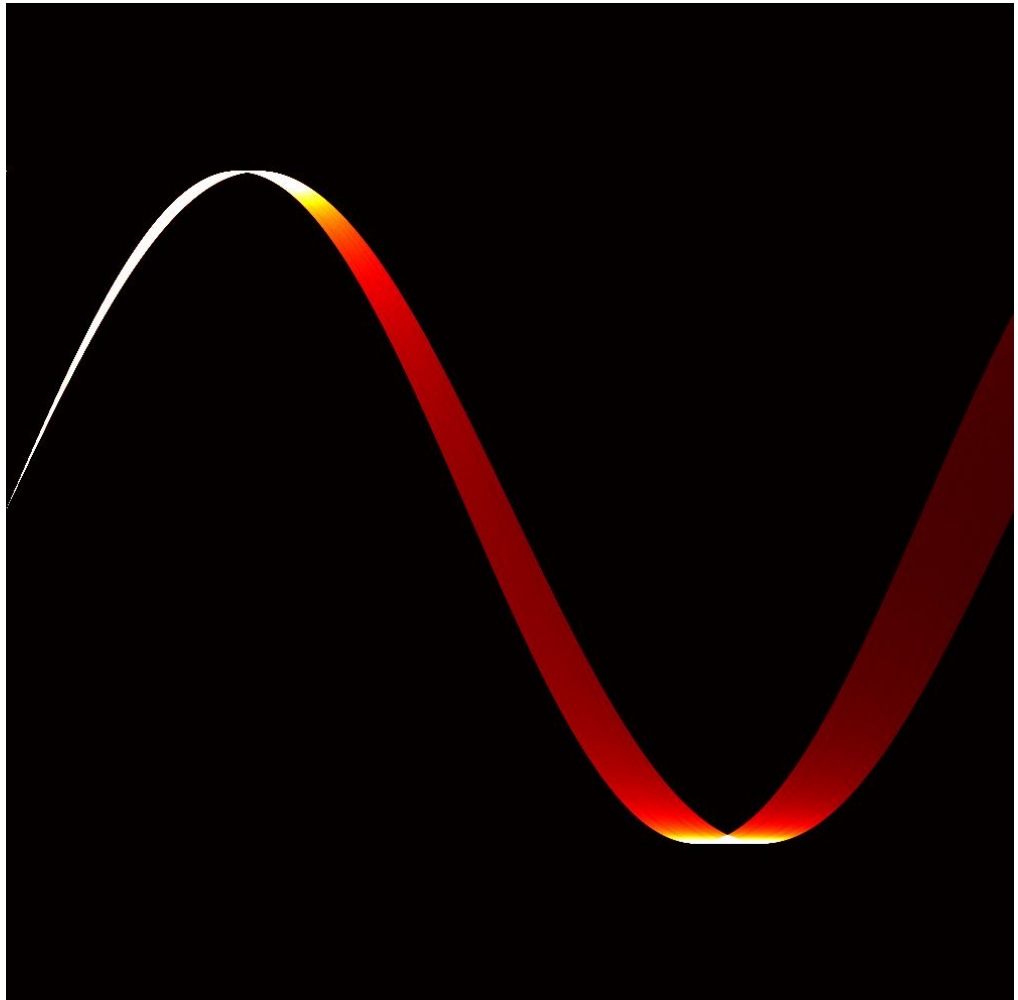
$$\overline{f} = f(\overline{x}) = f(2) = 2^2 = 4$$

$$f(x) \in [\underline{f}, \overline{f}] = [1, 4]$$

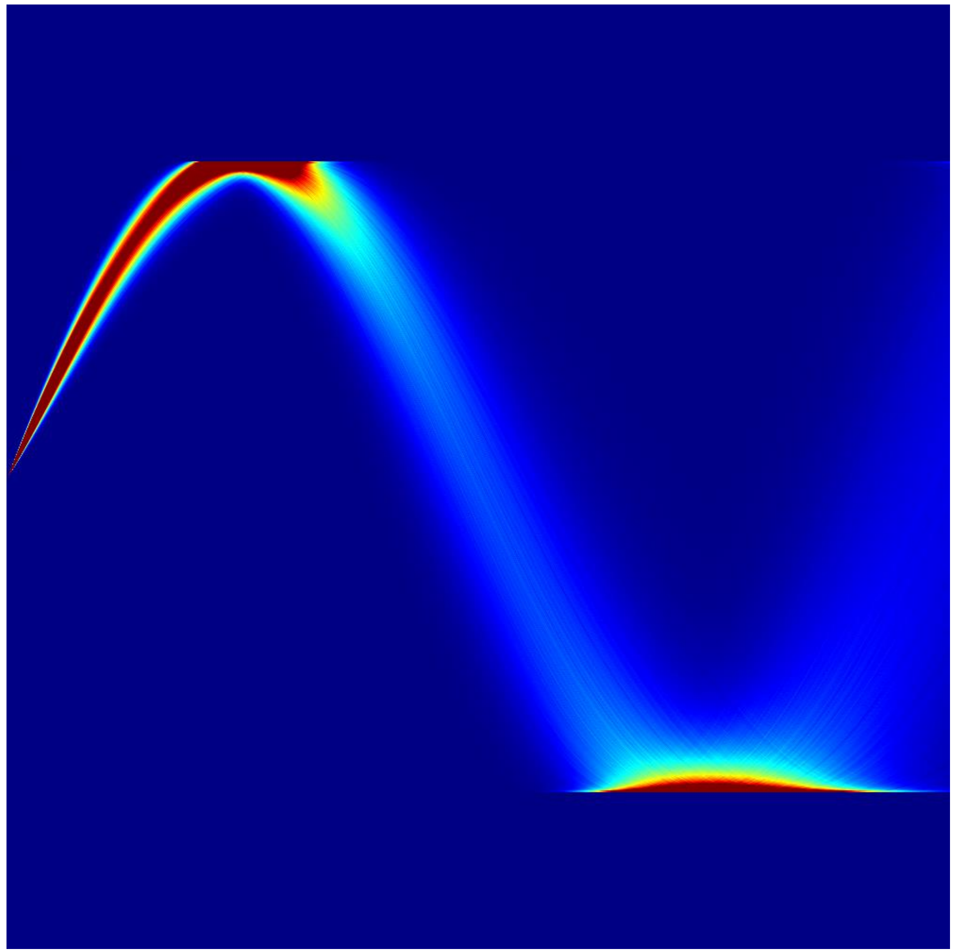
Interval solution of PDE

$$u(x,\mathbf{p}) = \{u(x,p) : p \in \mathbf{p}\}$$

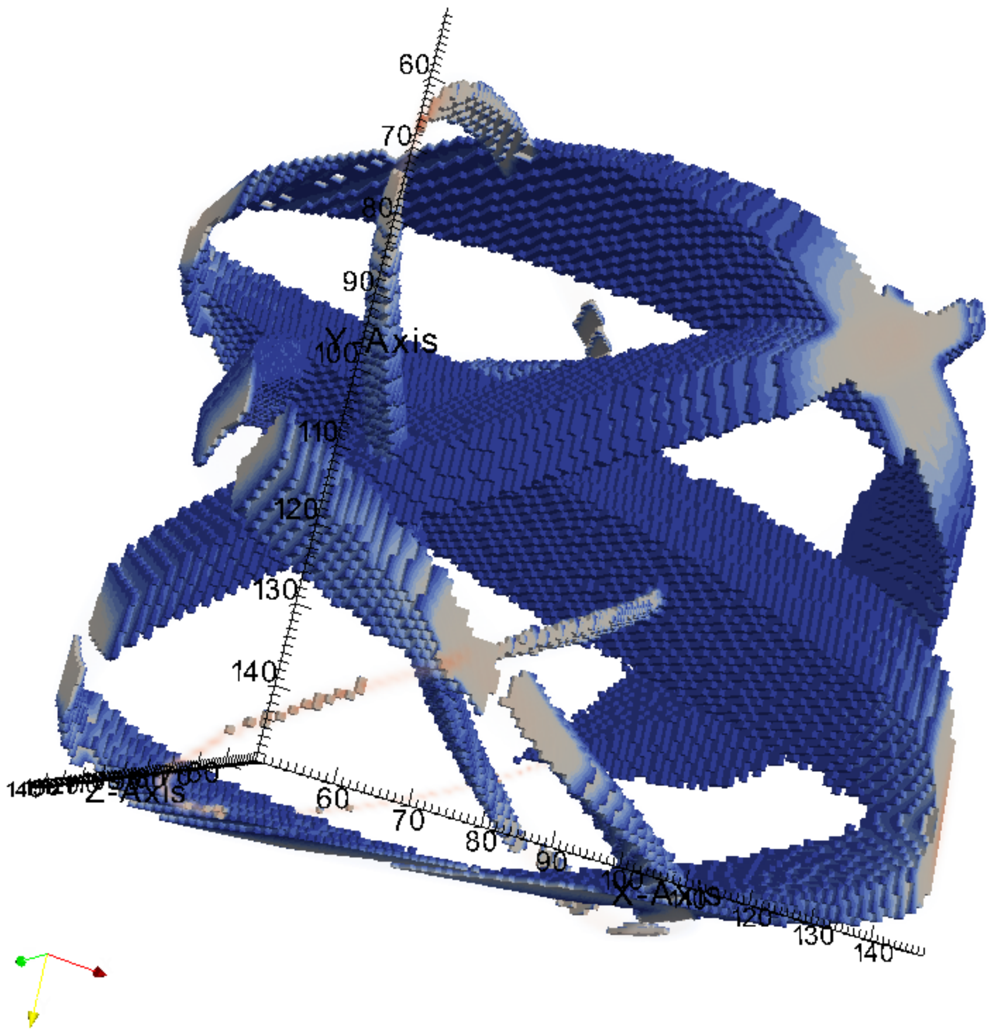
Set valued solution of ODE



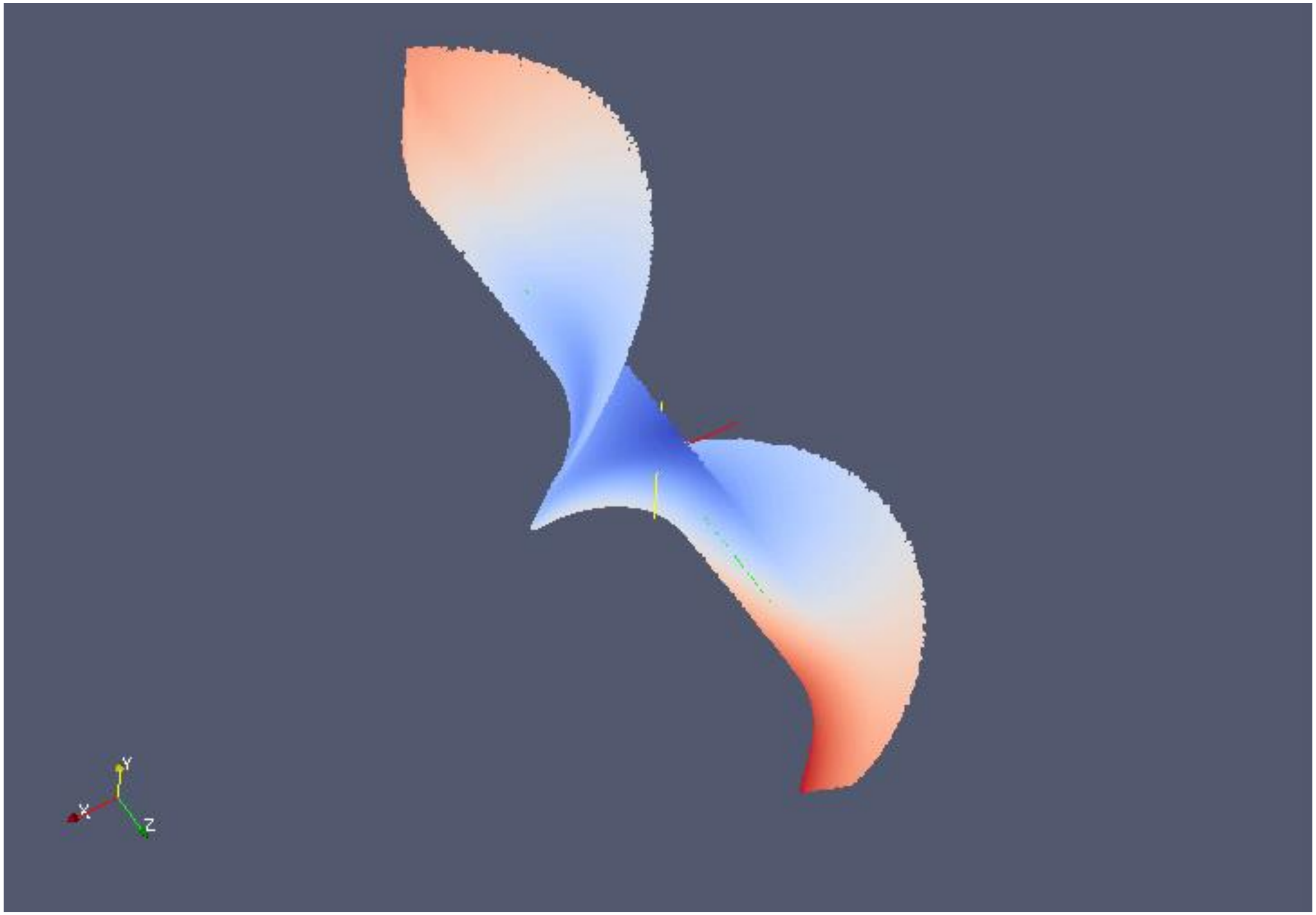
Random solution of ODE



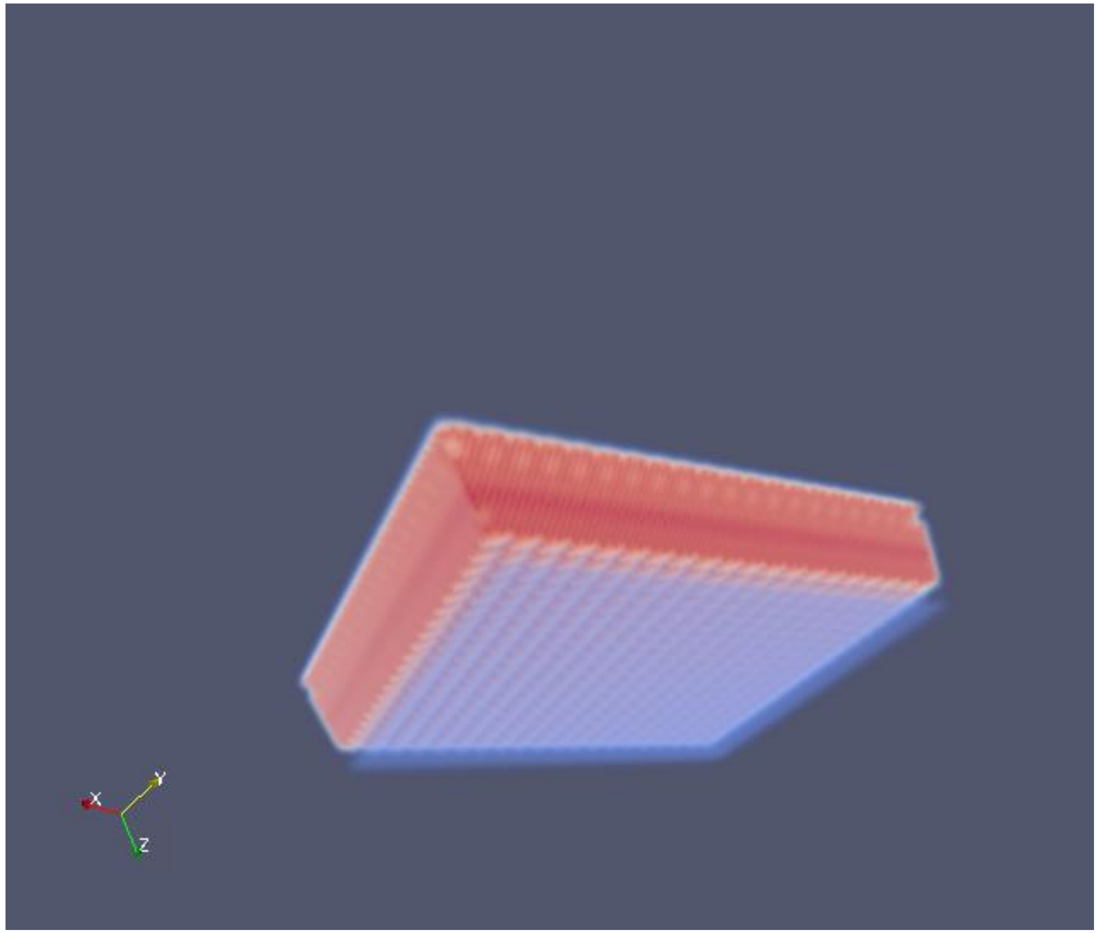
Set valued solution of ODE



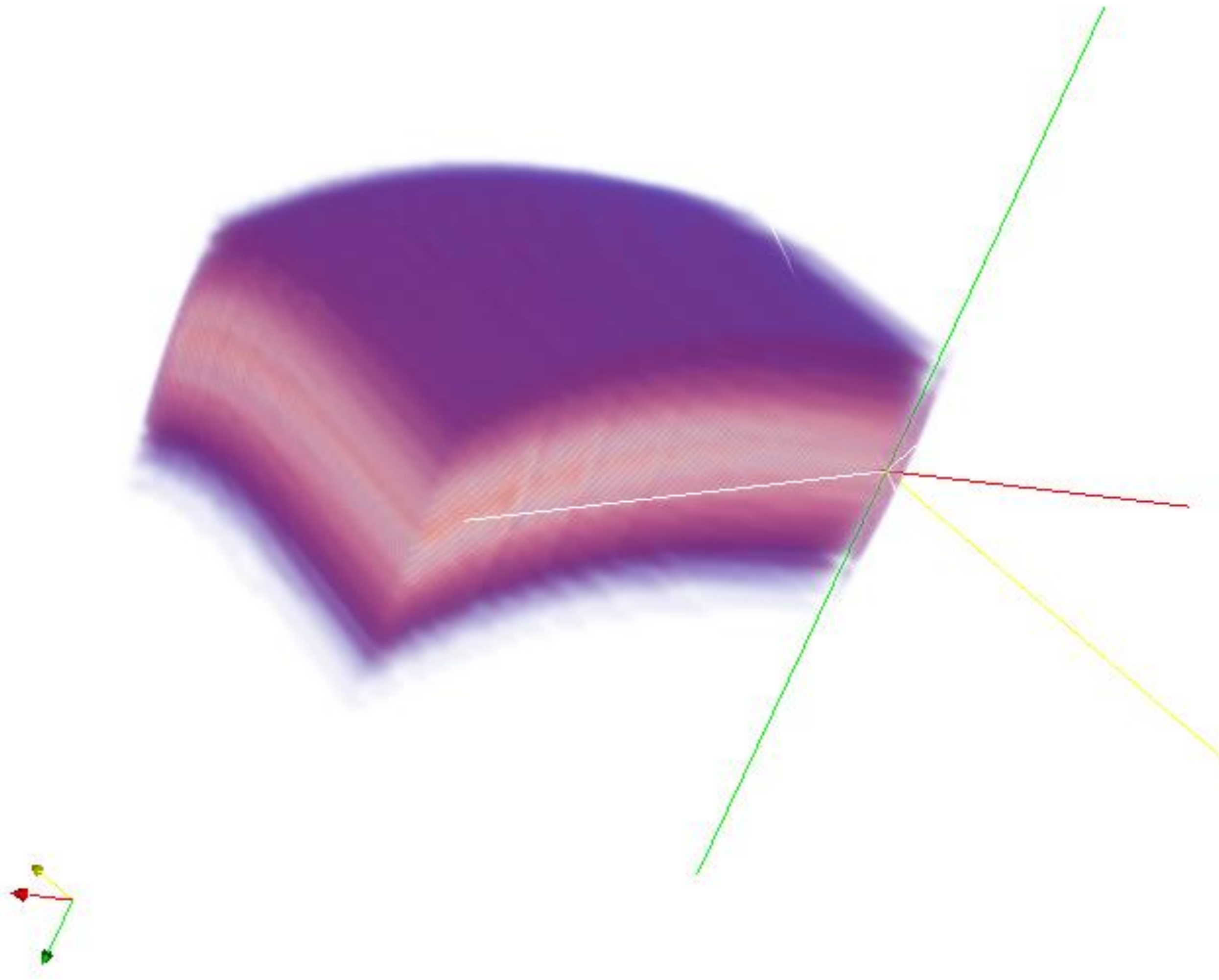
Solution set of algebraic equations



Set valued solution of PDE



Set valued solution of PDE



- Solution set of the equations with the interval parameters is not an interval

  - it is not necessary to reduce the set-valued solution the the interval solution.

$$\mathbf{u}(\mathbf{p}) = \square u(\mathbf{p})$$

- it is possible to use the set-valued solution directly in calculations.

Finite difference method

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{ij} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{ij} = f_{ij}$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f_{ij}$$

$$A(p)u = b(p)$$

Gradient method

$$A(p)\frac{\partial u}{\partial p}=\frac{\partial b(p)}{\partial p}-\frac{\partial A(p)}{\partial p}u$$

$$\underline{u}_i=\begin{cases}\min u_i\\ A(p)u=b(p)\\ p\in\mathbf{p}\end{cases},\quad \overline{u}_i=\begin{cases}\max u_i\\ A(p)u=b(p)\\ p\in\mathbf{p}\end{cases}$$

Optimization

$$p_i^{\min} = \arg\min \begin{cases} \min u_i \\ A(p)u = b(p) \\ p \in \mathbf{p} \end{cases}$$

$$p_i^{\max} = \arg\max \begin{cases} \max u_i \\ A(p)u = b(p) \\ p \in \mathbf{p} \end{cases}$$

New solution procedure

Stokes theorem

$$\int_{\Sigma} \nabla \times \mathbf{F} \cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r},$$

Butterfly lemma or **Zassenhaus lemma**

Lemma: Suppose (G,Ω) is a group with operators and A and C are subgroups. Suppose

$B \triangleleft A$ and $D \triangleleft C$

are stable subgroups. Then,

$(A \cap C)B/(A \cap D)B$ is isomorphic to $(A \cap C)D/(B \cap C)D.$

It is possible to use mathematical theorems in an automatic/computational way without human presence. (automated theorem proving)

It is possible to apply automatic theorem proving to the solution of the interval equations.

Automated theorem proving can be applied to the solution any scientific field. (biology, chemistry, physics …)

however this is not the topic of my research
look for example at: www.cyc.com
theory of everything …

It is possible to build a system who is able to use its own results in order to improve itself (self learning).

Examples

- 2*[1,2]*x-x=1
- (2*[1,2]-1)x=1
- ([2,4]-1)x=1
- [1,3]x=1
- x=1/[1,3]

- The system is able to show all intermediate steps of the calculations

  Look also at:
  http://www.wolframalpha.com

Example

$$\frac{du}{dx} + up = 0, u(0) = 0, p \in [1,2]$$

$$\frac{u_{i+1} - u_i}{\Delta x} + u_i p = 0, u_0 = 0, p \in [1,2]$$

$$u_{i+1} = u_i p \Delta x + u_i, u_0 = 0, p \in [1,2]$$

$$\underline{u}_i = \min\left\{u_i : u_{i+1} = u_i p \Delta x + u_i, u_0 = 0, p \in [1,2]\right\}$$
$$\overline{u}_i = \max\left\{u_i : u_{i+1} = u_i p \Delta x + u_i, u_0 = 0, p \in [1,2]\right\}$$

Example

- cats like food
- food=[milk< canned food<mice]

(example problem with the interval data)

- What kind of food do the cats like most?

- Answer: mice

Conclusions

- It is possible to solve efficiently the finite difference equations of computational mechanics with the interval parameters.
- In many cases it is possible to use general solution set of the system of interval equations instead of replacing it by the interval solution set.
- It is possible to apply automatic theorem proving in order to solve many problems in the theory of interval equations.
- It is possible to create self improving computational systems.