A Random Set Description of a Possibility Measure and Its Natural Extension

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Abstract— **The relationship is studied between possibility and necessity measures defined on arbitrary spaces, the theory of imprecise probabilities, and elementary random set theory. It is shown how special random sets can be used to generate normal possibility and necessity measures, as well as their natural extensions. This leads to interesting alternative formulas for the calculation of these natural extensions.**

Keywords— **Upper probability, upper prevision, coherence, natural extension, possibility measure, random sets.**

I. Introduction

POSSIBILITY measures were introduced by Zadeh [1] in 1978. In his view, these supremum preserving set functions are a mathematical representation of the information conveyed by typical affirmative statements in natural language. For recent discussions of this interpretation within the behavioural framework of the theory of imprecise probabilities, we refer to [2], [3], [4].

Supremum preserving set functions can also be found in the literature under a number of different guises. For instance, they appear in Shackle's logic of surprise [5], and were studied in a measure-theoretic context by Shilkret [6]. They also play the part of special limiting cases in Shafer's theory of belief functions [7]. In a number of recent papers [8], [9], [10], [11], we have studied in detail how possibility measures fit formally into the behavioural theory of imprecise probabilities [12]. Also relevant is the related work by Dubois and Prade [13], [14], and by Walley [3].

Here we discuss the relationship between possibility measures in the context of imprecise probabilities on the one hand, and random sets on the other hand.

We begin with a short overview of the relevant basic definitions. We work with a non-empty set Ω, called the *universe of discourse*. A *possibility measure* Π on $(\Omega, \wp(\Omega))$ is a mapping from the power class $\wp(\Omega)$ of Ω to the real unit interval $[0,1]$, which is supremum preserving in the following sense: for any family $(A_j \mid j \in J)$ of subsets of Ω,

$$\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j).$$

Such a possibility measure is completely determined by its *distribution* $\pi\colon \Omega \to [0,1]$, defined by $\pi(\omega) = \Pi(\{\omega\})$, $\omega \in \Omega$. Indeed, for any $A \in \wp(\Omega)$, $\Pi(A) = \sup_{\omega\in A} \pi(\omega)$. By definition, $\Pi(\emptyset) = 0$. Π is called *normal* if $\Pi(\Omega) = 1$.

We may associate with Π a dual *necessity measure* $\mathrm{N}\colon \wp(\Omega) \to [0,1]$, defined by $\mathrm{N}(A) = 1 - \Pi(\mathrm{co}A)$, $A \subseteq \Omega$, where co$A$ denotes the set-theoretic complement of A (relative to Ω). N is infimum preserving, and completely determined by its *distribution* $\nu\colon \Omega \to [0,1]$, defined by $\nu(\omega) = \mathrm{N}(\mathrm{co}\{\omega\}) = 1 - \pi(\omega)$, $\omega \in \Omega$. For any $A \in \wp(\Omega)$, $\mathrm{N}(A) = \inf_{\omega\in\mathrm{co}A} \nu(\omega)$. It follows from the definition that $\mathrm{N}(\Omega) = 1$, and we call N *normal* if Π is, i.e., if $\mathrm{N}(\emptyset) = 0$. For more details about the theory of possibility measures, we refer to [1], [15], [16], [17], [18], [19], [20].

As mentioned above, possibility and necessity measures can be incorporated into the behavioural theory of imprecise probabilities. Let us briefly describe how this is done. We limit ourselves here to definitions and results which are relevant in the context of this paper. For a detailed account of the theory of imprecise probabilities, we refer to the book by Walley [12].

The universe of discourse Ω can be interpreted as a possibility space, that is, the set of the mutually exclusive possible outcomes of a specific experiment. A *gamble* X on Ω is a bounded real-valued function on Ω, and can be interpreted as an uncertain reward. The set of all gambles on Ω is denoted by $\mathcal{L}(\Omega)$. An *event* A in Ω is a subset of Ω. The set of all events in Ω has already been given the notation $\wp(\Omega)$. We identify events with their characteristic functions, and interpret them as $0-1$-valued gambles. We also denote a *constant gamble* on Ω by the unique real value it assumes. The pointwise order on $\mathcal{L}(\Omega)$ is denoted by $\leq$, i.e., $X \leq Y$ if $(\forall\omega \in \Omega)(X(\omega) \leq Y(\omega))$.

An *upper prevision* $\overline{P}$ is a real-valued function on a set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. In order to identify its domain and possibility space, it is often denoted as $(\Omega, \mathcal{K}, \overline{P})$. The corresponding, or conjugate, *lower prevision* $(\Omega, -\mathcal{K}, \underline{P})$ is defined on the domain $-\mathcal{K} = \{-X\colon X \in \mathcal{K}\}$ as $\underline{P}(X) = -\overline{P}(-X)$, $X \in -\mathcal{K}$. In the behavioural context, $\overline{P}(X)$ can be interpreted as an infimum price for selling the gamble X, and $\underline{P}(X)$ as a supremum price for buying it. If $\mathcal{K}$ is in particular a class of events, then $\overline{P}$ is called an *upper probability*. The corresponding *lower probability* $\underline{P}$ is then defined on the set $\{\mathrm{co}A\colon A \in \mathcal{K}\}$ by $\underline{P}(A) = 1 - \overline{P}(\mathrm{co}A)$.

The set $\mathcal{L}(\Omega)$ is a linear space when provided with the pointwise addition of gambles and the pointwise scalar multiplication of gambles with real numbers. A linear functional P on $\mathcal{L}(\Omega)$ which is positive $(X \geq 0 \Rightarrow P(X) \geq 0)$ and has unit norm $(P(1) = 1)$ is called a *linear prevision* on $\mathcal{L}(\Omega)$. A linear prevision is a prevision, or fair price, in the sense of de Finetti [21]. Its restriction to $\wp(\Omega)$ is called a (finitely) *additive probability* on $\wp(\Omega)$. Note that $P(-X) = -P(X)$, $X \in \mathcal{L}(\Omega)$, which means that as an upper prevision, P is equal to the corresponding lower prevision, i.e., P is self-conjugate. The set of linear previsions on $\mathcal{L}(\Omega)$ is denoted by $\mathcal{P}(\Omega)$. If $\mathcal{K}$ is a subset of $\mathcal{L}(\Omega)$, a

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functional on $\mathcal{K}$ is called a linear prevision on $\mathcal{K}$ if it is the restriction to $\mathcal{K}$ of some linear prevision on $\mathcal{L}(\Omega)$. A similar definition is given for additive probabilities on arbitrary classes of events.

Given an upper prevision $(\Omega, \mathcal{K}, \overline{P})$, we define its set of dominated linear previsions $\mathcal{M}(\overline{P})$ as

$$\mathcal{M}(\overline{P}) = \{P \in \mathcal{P}(\Omega) \colon (\forall X \in \mathcal{K})(P(X) \leq \overline{P}(X))\}.$$

We say that $(\Omega, \mathcal{K}, \overline{P})$ *avoids sure loss* if $\mathcal{M}(\overline{P}) \neq \emptyset$, and is *coherent* if it avoids sure loss and

$$\overline{P}(X) = \sup\{P(X) \colon P \in \mathcal{M}(\overline{P})\}, \quad X \in \mathcal{K}.$$

When $(\Omega, \mathcal{K}, \overline{P})$ avoids sure loss, its *natural extension* $(\Omega, \mathcal{L}(\Omega), \overline{E})$ to $\mathcal{L}(\Omega)$ is defined as

$$\overline{E}(X) = \sup\{P(X) \colon P \in \mathcal{M}(\overline{P})\}, \quad X \in \mathcal{L}(\Omega).$$

It is the greatest coherent upper prevision that is dominated by $\overline{P}$ on its domain $\mathcal{K}$. $(\Omega, \mathcal{K}, \overline{P})$ is clearly coherent if and only if it coincides on its domain $\mathcal{K}$ with its natural extension. The natural extension $(\Omega, \mathcal{L}(\Omega), \underline{E})$ of the corresponding lower prevision $(\Omega, -\mathcal{K}, \underline{P})$ is defined by $\underline{E}(X) = -\overline{E}(-X) = \inf\{P(X) \colon P \in \mathcal{M}(\overline{P})\}$, $X \in \mathcal{L}(\Omega)$. Equivalent alternative definitions of avoiding sure loss, coherence and natural extension, with a *direct* behavioural interpretation, can be found in [12].

It has been shown elsewhere [3], [8], [9] that for a possibility measure Π on $(\Omega, \wp(\Omega))$ the following statements are equivalent: $(\Omega, \wp(\Omega), \Pi)$ avoids sure loss; $(\Omega, \wp(\Omega), \Pi)$ is a coherent upper probability; and Π is normal. It will therefore from now on be assumed that Π is normal. Its dual necessity measure is then the corresponding coherent lower probability. It can be verified[1] that their natural extensions to $\mathcal{L}(\Omega)$, respectively denoted by Π and N, are given by the following Riemann-Stieltjes integrals[2]:

$$\begin{aligned}\Pi(X) &= \int_{-\infty}^{+\infty} x \mathrm{d}\underline{F}_X(x) \\ &= \inf X + \int_{\inf X}^{\sup X} \sup\{\pi(\omega) \colon X(\omega) > x\} \mathrm{d}x \qquad (1)\end{aligned}$$

and

$$\begin{aligned}\mathrm{N}(X) &= \int_{-\infty}^{+\infty} x \mathrm{d}\overline{F}_X(x) \\ &= \inf X + \int_{\inf X}^{\sup X} \inf\{\nu(\omega) \colon X(\omega) < x\} \mathrm{d}x. \qquad (2)\end{aligned}$$

In these expressions, $\underline{F}_X$ and $\overline{F}_X$ are respectively the *lower* and *upper distribution functions* of the gamble X with respect to the pair Π and N, defined by, for any $y \in \mathbb{R}$:

$$\begin{aligned}\underline{F}_X(y) &= \mathrm{N}(\{\omega \in \Omega \colon X(\omega) \leq y\}) \\ \overline{F}_X(y) &= \Pi(\{\omega \in \Omega \colon X(\omega) \leq y\}).\end{aligned}$$

[1]The main point is that normal possibility measures are 2–alternating and normal necessity measures 2–monotone; the statement then follows from a result by Walley [12, Section 3.2.4].

[2]In the first expressions, the integration domains can be restricted to $[\inf[X] - \epsilon, \sup[X] + \delta]$ where $\epsilon > 0$ and $\delta > 0$, since X is bounded.

In what follows, we show that any possibility measure and any necessity measure can be constructed using a probability measure and a nested multivalued mapping. In particular, we thereby retrieve and at the same time refine a result proven by Dubois and Prade [13] for the special case that $\Omega = \mathbb{R}$. The line of reasoning we follow here draws from the ideas that lie at the basis of the theory of evidence [7], [22] and elementary random set theory [23].

At the same time, and more importantly, we show that this construction allows us to extend possibility and necessity measures in a straightforward way to upper and lower previsions which will turn out to be precisely their natural extensions, discussed above. In this way, we arrive at alternative formulas for the calculation of the natural extensions of possibility and necessity measures.

II. Multivalued mappings, gambles and events

Consider the real unit interval $[0,1]$ and a mapping Ψ from $[0,1]$ to $\wp(\Omega)$, also called a *multivalued mapping* from $[0,1]$ to Ω. We assume that Ψ is *antitone* in the following sense:

$$(\forall (x,y) \in [0,1]^2)(x \geq y \Rightarrow \Psi(x) \subseteq \Psi(y)), \qquad \text{(C1)}$$

which implies that $\Psi([0,1])$ is a *nested* collection of sets, or in other words, a chain of sets. We want every element of Ω to be contained in some $\Psi(x)$, $x \in [0,1]$, which due to (C1) amounts to:

$$\Psi(0) = \Omega. \qquad \text{(C2)}$$

For reasons that will become clear at the end of Section III, we do not exclude the existence of x in $[0,1]$ for which $\Psi(x) = \emptyset$. In order to deal with this, we define the set $\mathcal{E}_\Psi = \{x \in [0,1] \colon \Psi(x) = \emptyset\}$. Clearly, by (C1), $\mathcal{E}_\Psi$ is an up-set[3] of the complete chain $([0,1], \leq)$. If we define $\varepsilon_\Psi = \inf \mathcal{E}_\Psi$, then $\mathcal{E}_\Psi =]\varepsilon_\Psi, 1]$ or $\mathcal{E}_\Psi = [\varepsilon_\Psi, 1]$.

With any gamble X on Ω, we may associate two special gambles X^* and X_* on $[0,1]$, defined as follows:

$$X^*(x) = \begin{cases} \sup_{\omega \in \Psi(x)} X(\omega) & \text{if } x \in \mathrm{co}\mathcal{E}_\Psi \\ \theta^* & \text{if } x \in \mathcal{E}_\Psi \end{cases}$$

$$X_*(x) = \begin{cases} \inf_{\omega \in \Psi(x)} X(\omega) & \text{if } x \in \mathrm{co}\mathcal{E}_\Psi \\ \theta_* & \text{if } x \in \mathcal{E}_\Psi, \end{cases}$$

where θ^* and θ_* are real numbers to be determined shortly, and the complement 'co' is relative to the unit interval $[0,1]$. For $x \in \mathcal{E}_\Psi$, $\sup_{\omega \in \Psi(x)} X(\omega) = -\infty$ and $\inf_{\omega \in \Psi(x)} X(\omega) = +\infty$. Since we want X^* and X_* to be gambles on $[0,1]$, i.e., *bounded* $[0,1] - \mathbb{R}$-mappings, we try and remedy this by introducing θ^* and θ_*, which we determine by imposing extra conditions on X^* and X_*. First of all, we want that $(-X)^* = -X_*$, which is obviously equivalent[4] to $\theta^* + \theta_* = 0$. Also, for any event A in Ω, we find

[3]A subset A of a partially ordered set $(P, \leq)$ is an *up-set* if for any $a \in A$ and $p \in P$, $a \leq p \Rightarrow p \in A$, and a *down-set* if for any $a \in A$ and $p \in P$, $p \leq a \Rightarrow p \in A$ [24].

[4]This holds if $\mathcal{E}_\Psi \neq \emptyset$. If $\mathcal{E}_\Psi = \emptyset$, the introduction of θ^* and θ_* is not necessary.

in particular that

$$A^*(x) = \begin{cases} \theta^* & \text{if } x \in \mathcal{E}_\Psi \\ 0 & \text{if } x \notin \mathcal{E}_\Psi \text{ and } \Psi(x) \cap A = \emptyset \\ 1 & \text{if } x \notin \mathcal{E}_\Psi \text{ and } \Psi(x) \cap A \neq \emptyset, \end{cases}$$

and similarly

$$A_*(x) = \begin{cases} \theta_* & \text{if } x \in \mathcal{E}_\Psi \\ 0 & \text{if } x \notin \mathcal{E}_\Psi \text{ and } \Psi(x) \not\subseteq A \\ 1 & \text{if } x \notin \mathcal{E}_\Psi \text{ and } \Psi(x) \subseteq A. \end{cases}$$

We want A^* and A_* to correspond to events in $[0,1]$. θ^* and θ_* may therefore only assume the values 0 and 1. $\theta^* + \theta_* = 0$ then implies that $\theta^* = \theta_* = 0$, whence, with obvious notations,

$$A^* = \{x \in [0,1] \colon \Psi(x) \cap A \neq \emptyset\}$$
$$A_* = \{x \in [0,1] \colon \emptyset \neq \Psi(x) \subseteq A\}.$$

Note that $A_* \subseteq A^*$, $A_* \cap \mathcal{E}_\Psi = \emptyset$ and $A^* \cap \mathcal{E}_\Psi = \emptyset$. Moreover, $(\text{co}A)^* = \text{co}\mathcal{E}_\Psi \setminus A_*$, and $(\text{co}A)_* = \text{co}\mathcal{E}_\Psi \setminus A^*$, where the complement 'co' on the left hand sides is relative to Ω, and on the right hand sides relative to the real unit interval $[0,1]$. In particular, $\emptyset_* = \emptyset^* = \emptyset$ and $\Omega_* = \Omega^* = \text{co}\mathcal{E}_\Psi$.

The interpretation of these notions is rather straightforward. As we shall see further on, the multivalued mapping Ψ can be interpreted as a way to transmit information from $[0,1]$ to Ω. For a gamble X on Ω, X_* is the smallest gamble on $[0,1]$ compatible with X, and X^* the greatest. That $X_*(x) = X^*(x) = 0$ if $\Psi(x) = \emptyset$ assures that the gambles X_* and X^* are (behaviourally) neutral in those elements of $[0,1]$ that do not connect to elements of Ω.

We now investigate the Borel measurability of the events A^* and A_*, and the gambles X^* and X_*. For any event A in Ω, A^* is a down-set and $A_* \cup \mathcal{E}_\Psi$ an up-set of the chain $([0,1], \leq)$. Since we assumed that $\Psi(0) = \Omega$, it must also be that $A^* = \emptyset \Leftrightarrow A = \emptyset$, and $A_* = \text{co}\mathcal{E}_\Psi \Leftrightarrow A = \Omega$. Note that $A_* = \emptyset$ implies $(\text{co}A)^* = \text{co}\mathcal{E}_\Psi$, and that $A_* \neq \emptyset$ implies $A^* = \text{co}\mathcal{E}_\Psi$. Consider the following elements of $[0,1]$ for any $A \subseteq \Omega$:

$$\eta^*(A) = \sup A^* = \sup\{x \in [0,1] \colon \Psi(x) \cap A \neq \emptyset\}$$
$$\eta_*(A) = \inf(A_* \cup \mathcal{E}_\Psi) = \inf\{x \in [0,1] \colon \Psi(x) \subseteq A\}.$$

Note that both $\eta^*(A) \leq \varepsilon_\Psi$ and $\eta_*(A) \leq \varepsilon_\Psi$. We find that

$$\begin{aligned} A^* = [0, \eta^*(A)[\text{ or } A^* = [0, \eta^*(A)] \\ A_* \cup \mathcal{E}_\Psi = [\eta_*(A), 1] \text{ or } A_* \cup \mathcal{E}_\Psi =]\eta_*(A), 1]. \end{aligned} \tag{3}$$

To prove (3), remark that if $x < \eta^*(A)$, the characterisation of supremum on a chain tells us that there exists a y in A^* for which $x < y$ and therefore $x \in A^*$. A similar proof may be given for the second statement. Since we have seen above that $\mathcal{E}_\Psi =]\varepsilon_\Psi, 1]$ or $\mathcal{E}_\Psi = [\varepsilon_\Psi, 1]$, this implies that the sets A^* and A_* are subintervals of $[0,1]$ and therefore *Borel sets* on $[0,1]$.

In order to investigate the Borel measurability of the gambles X^* and X_*, we must for instance check whether, for every y in $\mathbb{R}$, the sets $\{X^* \leq y\} = \{x \in [0,1] \colon X^*(x) \leq y\}$ and $\{X_* \geq y\} = \{x \in [0,1] \colon X_*(x) \geq y\}$ are Borel sets on $[0,1]$. It may be verified that

$$\{X^* \leq y\} = \begin{cases} \{X \leq y\}_* & \text{if } y < 0 \\ \{X \leq y\}_* \cup \mathcal{E}_\Psi & \text{if } y \geq 0 \end{cases} \tag{4}$$

$$\{X_* \geq y\} = \begin{cases} \{X \geq y\}_* & \text{if } y > 0 \\ \{X \geq y\}_* \cup \mathcal{E}_\Psi & \text{if } y \leq 0, \end{cases} \tag{5}$$

where of course $\{X \geq y\} = \{\omega \in \Omega \colon X(\omega) \geq y\}$ and $\{X \leq y\} = \{\omega \in \Omega \colon X(\omega) \leq y\}$. Since in these expressions $\mathcal{E}_\Psi$, $\{X \leq y\}_*$ and $\{X \geq y\}_*$ are subintervals of $[0,1]$, and therefore Borel sets, we are led to the following proposition.

Proposition 1: For any gamble X on Ω, X^* and X_* are Borel measurable gambles on $[0,1]$. For any subset A of Ω, A^* and A_* are Borel measurable subsets of $[0,1]$.

III. Random sets, possibility and necessity measures

After these preliminary considerations, we are ready to proceed to the main topic of this paper. Consider a probability *measure* P_o on $([0,1], \mathcal{B}([0,1]))$, where $\mathcal{B}([0,1])$ is the σ-field of the Borel sets on $[0,1]$. In other words, P_o is a positive, *countably* additive set function defined on the set $\mathcal{B}([0,1])$, for which $P_o([0,1]) = 1$. Such a probability measure has a unique extension to a linear prevision $([0,1], \mathfrak{M}(\mathcal{B}([0,1])), E_{P_o})$, where $\mathfrak{M}(\mathcal{B}([0,1]))$ is the set of the $\mathcal{B}([0,1])$-measurable gambles on $[0,1]$, that is, the Borel measurable bounded $[0,1] - \mathbb{R}$-mappings [12]. As our notation suggests, this E_{P_o} is also the natural extension of the additive probability $([0,1], \mathcal{B}([0,1]), P_o)$ to the set of gambles $\mathfrak{M}(\mathcal{B}([0,1]))$. Moreover, for any Borel measurable gamble Y on $[0,1]$:

$$E_{P_o}(Y) = \int_{[0,1]} Y \mathrm{d}P_o \tag{6}$$

and the integral in this expression is the Lebesgue integral with respect to the measure P_o [12]. Note that the bounded mapping Y is always integrable on the compact set $[0,1]$. We shall assume that the measure P_o is *absolutely continuous*[5] with respect to the Lebesgue measure on $[0,1]$. Why this assumption is necessary, will become apparent in the proof of Lemma 3.

The introduction of the probability measure P_o on the measurable space $([0,1], \mathcal{B}([0,1]))$ allows us to formally interpret the multivalued mapping Ψ as a random variable, whose values are subsets of Ω. Ψ is therefore also called a *random set* [23], or a random subset of Ω.

We want to use the linear prevision E_{P_o} to construct a pair of coherent upper and lower previsions on $\mathcal{L}(\Omega)$, and consequently also a pair of coherent upper and lower probabilities on $\wp(\Omega)$. We shall assume that

$$P_o(\text{co}\mathcal{E}_\Psi) > 0, \text{ or equivalently, } P_o(\mathcal{E}_\Psi) < 1. \tag{C3}$$

[5]There seems to be no such requirement in the more restrictive context of [13], but we fail to see how it can be omitted even there (see also the proof of Lemma 3).

In other words, we *exclude* that the random set Ψ would be almost surely (P_o) empty, which would mean that $P_o(\{x \in [0,1] : \Psi(x) = \emptyset\}) = 1$. Proposition 1 then enables us to define an *upper prevision* $(\Omega, \mathcal{L}(\Omega), \Pi_\Psi)$ as follows. For any X in $\mathcal{L}(\Omega)$:

$$\begin{aligned} \Pi_\Psi(X) &= E_{P_o}(X^*)/P_o(\mathrm{co}\mathcal{E}_\Psi) \\ &= \int_{[0,1]} X^* \mathrm{d}P_o / P_o(\mathrm{co}\mathcal{E}_\Psi). \end{aligned} \tag{7}$$

Since we have made sure that $X_* = -(-X)^*$, and since E_{P_o} is a linear prevision on the linear space of the Borel measurable gambles on $[0,1]$, we find for the corresponding *lower prevision* $(\Omega, \mathcal{L}(\Omega), \mathrm{N}_\Psi)$, with $\mathrm{N}_\Psi(X) = -\Pi_\Psi(-X)$, that

$$\begin{aligned} \mathrm{N}_\Psi(X) &= E_{P_o}(X_*)/P_o(\mathrm{co}\mathcal{E}_\Psi) \\ &= \int_{[0,1]} X_* \mathrm{d}P_o / P_o(\mathrm{co}\mathcal{E}_\Psi). \end{aligned} \tag{8}$$

In what follows, we intend to study these upper and lower previsions in more detail.

For a start, we find the following expressions for the corresponding *upper* and *lower probability* of the event A in Ω that $\mathrm{N}_\Psi(A) = 1 - \Pi_\Psi(\mathrm{co}A)$,

$$\begin{aligned} \Pi_\Psi(A) &= P_o(A^*)/P_o(\mathrm{co}\mathcal{E}_\Psi) \\ &= P_o(\{x \in [0,1] : \Psi(x) \cap A \neq \emptyset\})/P_o(\mathrm{co}\mathcal{E}_\Psi) \end{aligned}$$

and

$$\begin{aligned} \mathrm{N}_\Psi(A) &= P_o(A_*)/P_o(\mathrm{co}\mathcal{E}_\Psi) \\ &= P_o(\{x \in [0,1] : \emptyset \neq \Psi(x) \subseteq A\})/P_o(\mathrm{co}\mathcal{E}_\Psi) \end{aligned}$$

These are instances of the more general formulas proposed by Dempster in his paper on upper and lower probabilities induced by a multivalued mapping [22].

In particular, we find that $\Pi_\Psi(\emptyset) = \mathrm{N}_\Psi(\emptyset) = 0$ and $\Pi_\Psi(\Omega) = \mathrm{N}_\Psi(\Omega) = 1$. Let us now show that the upper probability $(\Omega, \wp(\Omega), \Pi_\Psi)$ is a normal possibility measure, and that the lower probability $(\Omega, \wp(\Omega), \mathrm{N}_\Psi)$ is consequently a normal necessity measure. Consider any family $(A_j : j \in J)$ of subsets of Ω, then for any x in $[0,1]$:

$$\Psi(x) \cap \Big(\bigcup_{j \in J} A_j\Big) \neq \emptyset \Leftrightarrow (\exists j \in J)(\Psi(x) \cap A_j \neq \emptyset).$$

This implies that $(\bigcup_{j\in J} A_j)^* = \bigcup_{j \in J} A_j^*$ and therefore also $\eta^*(\bigcup_{j\in J} A_j) = \sup_{j\in J} \eta^*(A_j)$. If we combine this with (3), we find that $(\bigcup_{j\in J} A_j)^* = [0, \sup_{j\in J} \eta^*(A_j)[$ or $(\bigcup_{j\in J} A_j)^* = [0, \sup_{j\in J} \eta^*(A_j)]$. When we invoke Lemma 3, we find the following result.

Theorem 2: The restriction of Π_Ψ to $\wp(\Omega)$ is a normal possibility measure and the restriction of N_Ψ to $\wp(\Omega)$ a normal necessity measure on $(\Omega, \wp(\Omega))$. For their respective distributions π_Ψ and ν_Ψ we find, for any ω in Ω:

$$\begin{aligned} \pi_\Psi(\omega) &= P_o(\{x \in [0,1] : \omega \in \Psi(x)\})/P_o(\mathrm{co}\mathcal{E}_\Psi) \\ \nu_\Psi(\omega) &= P_o(\{x \in [0,1] : \omega \notin \Psi(x) \neq \emptyset\})/P_o(\mathrm{co}\mathcal{E}_\Psi). \end{aligned}$$

The upper probability $(\Omega, \wp(\Omega), \Pi_\Psi)$ and the lower probability $(\Omega, \wp(\Omega), \mathrm{N}_\Psi)$ are therefore coherent.

Lemma 3: Consider a family $(x_j : j \in J)$ of elements of $[0,1]$ and an element x of $[0,1]$.
1. $P_o([0,x[) = P_o([0,x])$.
2. $P_o([0, \sup_{j\in J} x_j]) = \sup_{j\in J} P_o([0, x_j])$.

Proof: The proof of the first statement is immediate, since P_o is by assumption absolutely continuous with respect to the Lebesgue measure on $[0,1]$. Let us therefore prove the second statement. Since P_o is increasing, it need only be shown that $\sup_{j\in J} P_o([0,x_j]) \geq P_o([0, \sup_{j\in J} x_j])$. If $\sup_{j\in J} x_j$ belongs to the family $(x_j : j \in J)$, the proof is immediate. Let us therefore assume that $x_i < \sup_{j\in J} x_j$, $i \in J$, which implies that there exists a strictly increasing sequence $(x_n^*)_{n\in\mathbb{N}}$ in the family $(x_j : j \in J)$ which converges to $\sup_{j\in J} x_j$. As a consequence, the strictly increasing sequence $([0, x_n^*])_{n\in\mathbb{N}}$ converges to $\bigcup_{n\in\mathbb{N}}[0, x_n^*] = [0, \sup_{j\in J} x_j[$, and therefore, taking into account the well-known limit properties of a measure:

$$\begin{aligned} \sup_{j\in J} P_o([0,x_j]) &\geq \sup_{n\in\mathbb{N}} P_o([0,x_n^*]) \\ &= P_o\Big(\bigcup_{n\in\mathbb{N}} [0,x_n^*]\Big) = P_o([0, \sup_{j\in J} x_j[). \end{aligned}$$

This proves the second statement, also taking into account the first. ■

Dubois and Prade [13] have proven a similar result in the special case that the sets $\Psi(x)$, $x \in]0,1]$, are bounded and closed real intervals. In the case of a finite universe of discourse Ω, the essence of this result was already known to Shafer [7].

Remark 1: If we were to start out with a coherent upper probability $\overline{P}_o$, defined on a suitably large collection of subsets of $[0,1]$, rather than with the probability measure P_o, we would for instance still be able to define an upper probability $\overline{P}_\Psi$ on the universe of discourse Ω through the formula $\overline{P}_\Psi(A) = \overline{P}_o(A^*)/\overline{P}_o(\mathrm{co}\mathcal{E}_\Psi)$, provided of course that $\overline{P}_o(\mathrm{co}\mathcal{E}_\Psi) > 0$. It is clear from the previous discussion that $\overline{P}_\Psi$ will be *maxitive* in the sense that $\overline{P}_\Psi(A \cup B) = \max\{\overline{P}_\Psi(A), \overline{P}_\Psi(B)\}$. Whether $\overline{P}_\Psi$ will be a possibility measure, depends on the continuity properties of the upper probability $\overline{P}_o$. As an example, we have just shown that if $\overline{P}_o$ is a probability measure, its absolute continuity with respect to Lebesgue measure is sufficient to guarantee that $\Pi_\Psi = \overline{P}_\Psi$ is not only maxitive but also a possibility measure. If Ω is finite, the coherence of $\overline{P}_o$ is sufficient for $\overline{P}_\Psi$ to be a possibility measure.

In what follows, we shall call the couple (P_o, Ψ) a *random set representation* of the possibility measure Π_Ψ and the necessity measure N_Ψ on $(\Omega, \wp(\Omega))$.

We now proceed to show that *every normal possibility and necessity measure* on $(\Omega, \wp(\Omega))$ can be obtained in this way, i.e., have a random set representation! Indeed, given any normal possibility measure Π on $(\Omega, \wp(\Omega))$, define the multivalued mapping Ψ as follows: for any x in $[0,1]$, $\Psi(x) = \{\pi \geq x\} = \{\omega \in \Omega : \pi(\omega) \geq x\}$, where π is the distribution of Π. This Ψ clearly satisfies conditions (C1) and (C2). Furthermore, for any ω in Ω, $\{\omega\}^* = [0, \pi(\omega)]$. If we therefore let P_o be the Lebesgue measure λ on $[0,1]$, its is clear that $P_o(\{\omega\}^*) = \pi(\omega)$. Finally, since Π is normal, $\sup_{\omega\in\Omega}\pi(\omega) = 1$, whence $\mathcal{E}_\Psi = \emptyset$ or $\mathcal{E}_\Psi = \{1\}$, according to whether the supremum 1 of π is reached or not. In any case, $P_o(\mathrm{co}\mathcal{E}_\Psi) = 1$ and (C3) is satisfied. This proves our assertion. The couple $(\lambda, \{\pi \geq \cdot\})$ will be called the *standard* random set representation of the possibility measure Π and its dual necessity measure N (see also [25], [26]).

We want to stress that for this choice of Ψ, $\mathcal{E}_\Psi$ is not necessarily empty! In the case $\mathcal{E}_\Psi = \{1\}$, π does not reach its supremum in any of the points of its domain. The distribution of Π is then called *non-modal*. It therefore turns out that we had to allow $\mathcal{E}_\Psi \neq \emptyset$ in order to be able to incorporate into the random set model *normal* possibility measures with *non-modal* distributions.

IV. Random sets and natural extension

Let us now use this information to derive a formula for the calculation of $\Pi_\Psi(X)$ and $\mathrm{N}_\Psi(X)$, $X \in \mathcal{L}(\Omega)$. Walley has shown [12, Section 3.2.4] that the expression (6) for the expectation $E_{P_o}(Y)$ of a Borel measurable gamble Y on $[0,1]$ can also be written as a Riemann-Stieltjes integral:

$$
\begin{aligned}
E_{P_o}(Y) &= \int_{[0,1]} Y \mathrm{d}P_o \\
&= \int_{-\infty}^{+\infty} x \mathrm{d}F_Y(x) \\
&= \inf[Y] + \int_{\inf[Y]}^{\sup[Y]} 1 - F_Y(x)\mathrm{d}x
\end{aligned}
$$

where $F_Y(x) = P_o(\{Y \leq x\})$. Since for any gamble X on Ω, the corresponding gamble X^* on $[0,1]$ is Borel measurable, this allows us to rewrite Eq. (7) as

$$
\begin{aligned}
\Pi_\Psi(X) &= \int_{[0,1]} X^* \mathrm{d}P_o / P_o(\mathrm{co}\mathcal{E}_\Psi) \\
&= \int_{-\infty}^{+\infty} x \mathrm{d}F_{X^*}(x)/P_o(\mathrm{co}\mathcal{E}_\Psi)
\end{aligned} \tag{9}
$$

where for any x in $\mathbb{R}$, $F_{X^*}(x) = P_o(\{X^* \leq x\})$. Now, taking into account (4) and (C3), we find for any y in $\mathbb{R}$,

$$
\begin{aligned}
F_{X^*}(x) &= P_o(\{X^* \leq x\}) \\
&= \begin{cases} P_o(\{X \leq x\}_*) & \text{if } x < 0 \\ P_o(\{X \leq x\}_* \cup \mathcal{E}_\Psi) & \text{if } x \geq 0 \end{cases} \\
&= \begin{cases} P_o(\mathrm{co}\mathcal{E}_\Psi)\mathrm{N}_\Psi(\{X \leq x\}) & \text{if } x < 0 \\ P_o(\mathrm{co}\mathcal{E}_\Psi)\mathrm{N}_\Psi(\{X \leq x\}) + P_o(\mathcal{E}_\Psi) & \text{if } x \geq 0 \end{cases} \\
&= P_o(\mathrm{co}\mathcal{E}_\Psi) \cdot \begin{cases} \underline{F}_X(x) & \text{if } x < 0 \\ \underline{F}_X(x) + \dfrac{P_o(\mathcal{E}_\Psi)}{P_o(\mathrm{co}\mathcal{E}_\Psi)} & \text{if } x \geq 0, \end{cases}
\end{aligned} \tag{10}
$$

If we substitute (10) into (9), we find, taking into account well-known results from the theory of Riemann-Stieltjes integration [27]:

$$
\begin{aligned}
\Pi_\Psi(X) &= \int_{-\infty}^{+\infty} x \mathrm{d}\underline{F}_X(x) + 0 \cdot \frac{P_o(\mathcal{E}_\Psi)}{P_o(\mathrm{co}\mathcal{E}_\Psi)} \\
&= \int_{-\infty}^{+\infty} x \mathrm{d}\underline{F}_X(x),
\end{aligned} \tag{11}
$$

and consequently also

$$
\mathrm{N}_\Psi(X) = \int_{-\infty}^{+\infty} x \mathrm{d}\overline{F}_X(x), \tag{12}
$$

where $\overline{F}_X$ is the upper distribution function of X associated with the pair Π_Ψ and N_Ψ. In other words, using the probability measure P_o and the nested multivalued mapping Ψ, we are able to construct in a natural way not only possibility and necessity measures, but *also their natural extensions!* As a corollary, we find that the upper and lower previsions $(\Omega, \mathcal{L}(\Omega), \Pi_\Psi)$ and $(\Omega, \mathcal{L}(\Omega), \mathrm{N}_\Psi)$ are coherent.

The right-hand sides of (11) and (12) are precisely the expressions Dempster has suggested for calculating what he calls the upper and lower expectation of X associated with the upper and lower probabilities Π_Ψ and N_Ψ [22].

Remark 2: The argument relating the natural extension of Π_Ψ in a gamble X to the expectation $E_{P_o}(X^*)$ does not depend on the properties of Ψ, nor on the fact that Ψ is defined on $[0,1]$. In particular, this means that this connection is more generally valid for completely alternating upper probabilities (or plausibility functions) rather than just for possibility measures. It also remains valid if we replace the probability measure P_o by a 2–alternating upper probability: the crucial point is the induced upper probability is then also 2–alternating, and that natural extension for 2–alternating upper probabilities can be calculated using a Choquet integral.

This result, together with the argument presented at the end of the previous section, also allows us to give an alternative expression for the natural extensions of a possibility measure Π and its dual necessity measure N on $(\Omega, \wp(\Omega))$. Let π be the distribution of Π. For the standard random set representation $(P_o, \Psi) = (\lambda, \{\pi \geq \cdot\})$ of Π and N, we know that $P_o(\mathrm{co}\mathcal{E}_\Psi) = 1$, whence, if we also denote by Π the natural extension of Π, $\Pi(X) = \int_{[0,1]} X^* \mathrm{d}\lambda$, where in particular for $x \in [0,1[$, $X^*(x) = \sup\{X(\omega) : \pi(\omega) \geq x\}$. Since X^* is decreasing on $[0,1[$, it is discontinuous in at most a countable number of elements of $[0,1]$, and therefore the Lebesgue integral is equal to the corresponding Riemann integral [28]:

$$
\begin{aligned}
\Pi(X) &= \int_0^1 \sup\{X(\omega) : \pi(\omega) \geq x\} \mathrm{d}x \\
&= \int_0^1 \sup\{X(\omega) : \pi(\omega) > x\} \mathrm{d}x.
\end{aligned} \tag{13}
$$

The second equality holds because the two integrands may only differ in their points of discontinuity, which are at most countable in number. Similarly, if we also denote by N the natural extension of N:

$$\begin{aligned} \mathrm{N}(X) &= \int_0^1 \inf\{X(\omega)\colon \pi(\omega) \geq x\}\mathrm{d}x \\ &= \int_0^1 \inf\{X(\omega)\colon \pi(\omega) > x\}\mathrm{d}x \end{aligned} \tag{14}$$

or in terms of the distribution ν of N:

$$\begin{aligned} \mathrm{N}(X) &= \int_0^1 \inf\{X(\omega)\colon \nu(\omega) \leq x\}\mathrm{d}x \\ &= \int_0^1 \inf\{X(\omega)\colon \nu(\omega) < x\}\mathrm{d}x. \end{aligned}$$

When we compare these expressions with (1) and (2), the symmetry (exchangeability) between π and ν on the one hand, and X on the other hand, is striking.

Formulas (13) and (14) have been found independently by Peter Walley and were published recently in [3]. The proof he has found is different from the one given here, and does not rely on the link between possibility measures and random sets, which has a central part in this paper. It was first published in [29]. An even more direct proof of (13), based on a suggestion by Hugo Janssen, runs as follows. It is obvious that $\Pi(X)$ as given by (1) is the *Choquet integral* of X with respect to the set function Π, which according to an important result by Denneberg [30] is equal to

$$\Pi(X) = \int_0^{\Pi(\Omega)} \check{G}_{\Pi,X}(t)\mathrm{d}t = \int_0^1 \check{G}_{\Pi,X}(t)\mathrm{d}t.$$

In this expression,

$$G_{\Pi,X}(x) = \Pi(\{\omega\colon X(\omega) \geq x\}), \quad x \in [\inf X, \sup X]$$

is the *decreasing* distribution function of X with respect to Π. Let us denote the interval $[\inf X, \sup X]$ by $\mathbb{X}$. The *pseudo-inverse* $\check{G}_{\Pi,X}$ of $G_{\Pi,X}$ can for our purposes be defined as a $[0,1] - \mathbb{X}$-mapping, with

$$\check{G}_{\Pi,X}(t) = \sup\{x \in \mathbb{X}\colon G_{\Pi,X}(x) > t\}), \quad t \in [0,1],$$

where here and in what follows, we set $\sup\emptyset = \inf X$ for suprema associated with the chain $\mathbb{X}$. Note that, consequently, $\check{G}_{\Pi,X}(1) = \inf X$. Moreover, for any $t \in [0,1[$, the set $\{x \in \mathbb{X}\colon G_{\Pi,X}(x) > t\}$ is nonempty, since Π is normal. The proof consists in showing that $\check{G}_{\Pi,X}(t) = \sup\{X(\omega)\colon \pi(\omega) > t\}$, $t \in [0,1]$.

First of all, let us prove that for any t in $[0,1]$,

$$\check{G}_{\Pi,X}(t) = \sup\{y \in \mathbb{X}\colon \Pi(\{\omega\colon X(\omega) > y\}) > t\}.$$

It is obvious that the equality holds for $t = 1$. We therefore concentrate on $t \in [0,1[$. Clearly,

$$\check{G}_{\Pi,X}(t) \geq \sup\{y \in \mathbb{X}\colon \Pi(\{\omega\colon X(\omega) > y\}) > t\}.$$

To prove that the reverse inequality holds, consider any x in $\mathbb{X}$ such that $\Pi(\{\omega\colon X(\omega) \geq x\}) > t$. There always exists such an x since $t < 1$. For any $\epsilon > 0$, $\Pi(\{\omega\colon X(\omega) > x - \epsilon\}) \geq \Pi(\{\omega\colon X(\omega) \geq x\}) > t$, whence $x - \epsilon \leq \sup\{y \in \mathbb{X}\colon \Pi(\{\omega\colon X(\omega) > y\}) > t\}$. Since this holds for arbitrary strictly positive ϵ, we may conclude that $x \leq \sup\{y \in \mathbb{X}\colon \Pi(\{\omega\colon X(\omega) > y\}) > t\}$, and therefore also that

$$\check{G}_{\Pi,X}(t) \leq \sup\{y \in \mathbb{X}\colon \Pi(\{\omega\colon X(\omega) > y\}) > t\}.$$

Finally, it is readily verified that for any $x \in \mathbb{X}$ and $t \in [0,1]$, $\Pi(\{\omega\colon X(\omega) > x\}) > t$ if and only if $\sup\{X(\omega)\colon \pi(\omega) > t\} > x$, whence indeed $\check{G}_{\Pi,X}(t) = \sup\{X(\omega)\colon \pi(\omega) > t\}$.

V. Conclusion

We have shown that any possibility and necessity measure can be constructed using a probability measure and a multivalued mapping (a random set). At the same time, and more importantly, we have proven that their natural extensions can be obtained in a similar way. This course of reasoning has provided us with alternative formulas for the calculation of these natural extensions.

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References

[1] L. A. Zadeh, "Fuzzy sets as a basis for a theory of possibility," *Fuzzy Sets and Systems*, vol. 1, pp. 3–28, 1978.

[2] G. de Cooman, "Describing linguistic information in a behavioural framework: possible or not?," in *Intelligent Systems: A Semiotic Perspective (Proceedings of the 1996 International Multidisciplinary Conference, Gaithersburg, MD, 20–23 October 1996)*, J. Albus, A. Meystel, and R. Quintero, Eds., pp. 141–150. 1996.

[3] P. Walley, "Measures of uncertainty in expert systems," *Artificial Intelligence*, vol. 83, pp. 1–58, 1996.

[4] P. Walley and G. de Cooman, "A behavioural model for linguistic uncertainty," in *Computing with Words*, P. P. Wang, Ed. John Wiley & Sons, New York, 1998, Accepted for publication.

[5] G. L. S. Shackle, *Decision, Order and Time in Human Affairs*, Cambridge University Press, Cambridge, 1961.

[6] N. Shilkret, "Maxitive measures and integration," *Indagationes Mathematicae*, vol. 33, pp. 109–116, 1971.

[7] G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, NJ, 1976.

[8] G. de Cooman and D. Aeyels, "On the coherence of supremum preserving upper previsions," in *Proceedings of IPMU '96 (Sixth International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, Granada, Spain, 1–5 July 1996)*, 1996, pp. 1405–1410.

[9] G. de Cooman and D. Aeyels, "Supremum preserving upper probabilities," *Information Sciences*, 1999, In the press.

[10] G. de Cooman, "Integration in possibility theory," in *Fuzzy Measures and Integrals – Theory and Applications*, M. Grabisch, T. Murofushi, and M. Sugeno, Eds. Physica-Verlag (Springer), Heidelberg, 1999, Accepted for publication.

[11] P. Walley and G. de Cooman, "Coherence of rules for defining conditional possibility," *International Journal of Approximate Reasoning*, vol. 21, pp. 63–107, 1999.

[12] P. Walley, *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, London, 1991.

[13] D. Dubois and H. Prade, "The mean value of a fuzzy number," *Fuzzy Sets and Systems*, vol. 24, pp. 279–300, 1987.

[14] D. Dubois and H. Prade, "When upper probabilities are possibility measures," *Fuzzy Sets and Systems*, vol. 49, pp. 65–74, 1992.

[15] G. de Cooman, "Possibility theory I: the measure- and integral-theoretic groundwork," *International Journal of General Systems*, vol. 25, pp. 291–323, 1997.

[16] G. de Cooman, "Possibility theory II: conditional possibility," *International Journal of General Systems*, vol. 25, pp. 325–351, 1997.

[17] G. de Cooman, "Possibility theory III: possibilistic independence," *International Journal of General Systems*, vol. 25, pp. 353–371, 1997.

[18] G. de Cooman and E. E. Kerre, "Possibility and necessity integrals," *Fuzzy Sets and Systems*, vol. 77, pp. 207–227, 1996.

[19] D. Dubois and H. Prade, *Théorie des possibilités*, Masson, Paris, 1985.

[20] D. Dubois and H. Prade, *Possibility Theory*, Plenum Press, New York, 1988.

[21] B. de Finetti, *Theory of Probability*, vol. 1, John Wiley & Sons, Chichester, 1974, English Translation of *Teoria delle Probabilità*.

[22] A. P. Dempster, "Upper and lower probabilities induced by a multivalued mapping," *Annals of Mathematical Statistics*, vol. 38, pp. 325–339, 1967.

[23] G. Matheron, *Random Sets and Integral Geometry*, John Wiley & Sons, New York, 1975.

[24] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 1990.

[25] S. Chanas and M. Nowakowski, "Single value simulation of fuzzy variable," *Fuzzy Sets and Systems*, vol. 25, pp. 43–57, 1988.

[26] I. R. Goodman, "Some new results concerning random sets and fuzzy sets," *Information Sciences*, vol. 34, pp. 93–113, 1984.

[27] T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, Reading, MA, 1975.

[28] R. B. Ash, *Real Analysis and Probability*, Academic Press, New York, 1972.

[29] P. Walley, "Statistical inferences based on a second-order possibility distribution," *International Journal of General Systems*, vol. 26, pp. 337–383, 1997.

[30] D. Denneberg, *Non-Additive Measure and Integral*, Kluwer Academic, Dordrecht, 1994.

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