Granular Fuzzy Sets: A View from Rough Set and Probability Theories

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Abstract

To each point c in a base space C (universe of discourse), we associate a set U_C with a probability μ. Let $U=\cup_C U_C$ and be called the total space (a context or background). Then a subset $X\subseteq U$ defines a membership function $F_X(c)=\mu(X\cap U_C)$ on C, called a granular membership function. The main results are: (1) For a given C, there is a product granular structure such that all fuzzy sets with s- and t-norms can be represented by granular fuzzy sets. As a consequence, (2) if U is finite and all members in U_C are truly indiscernible, then *rough set theory of U induces fuzzy set theory on C*. (3) A somewhat surprised conclusion is that in the classes of traditional fuzzy sets, *the grade can not be interpreted as a probability.*

Keywords: Fuzzy Set, Rough Set, Rough Granular Structure, Probability.

1. Introduction

1.1. Background

Fuzzy set theory [25] models the uncertainty by real valued functions, called membership functions. In its original formulation, the grade is not formally interpreted; a fuzzy set is solely defined by its membership function. [26] (pp.12).

Since the probability is the most common notion of uncertainty, there is a temptation to interpret the grade as a probability. Unfortunately, it is often done in an improper manner. This paper is motivated by the intent of settling this problem

Let us fixe some notations. Let C be the universe of discourse, called a *base spac*e, on which a membership function defines a fuzzy set.

1.2 The Hidden Assumptions of a Probability

There is a common misconception that a real value between 0 and 1 is a probability; this is far from the truth. To talk about a probability, one needs a probability space; a probability is a measure of an event (a subset); see appendix.

To apply such a notion of probability to the grades of memberships, we need to associate a probability space, denoted by U_C, to each point c in the base space. The total union of such probability spaces, U_C's, will be called the total space U. We will represent the association by a map $\Pi: U \rightarrow C$, where each member of U_C is mapped to c. The triple (U, Π, C), called a granular structure, is the hidden context for talking about the grade as probability.

1.3 Rough Set Theory

Rough set theory, proposed by Zdzislaw Pawlak [20] and independently by T. Lee [5], is essentially a theory of relational instances (snapshots of relational databases). The primary notion is a partition R on a universe U, which classifies the members into mutually disjoint equivalence classes (granules). It naturally form a granular structure (U, Π, U/R) that consists of the universe U, the quotient set U/R (the set of all equivalence classes), and the natural projection $\Pi: U \rightarrow U/R$.

In the rough set environment, U and C =U/R are finite. Any subset X of U defines a membership function on C as follows: $F_X(c) = |X\cap U_C|/|U_C|$. The grade, $F_X(c) = 1/2$, takes a very elementary interpretation, that is, it specifies exactly one half (by counting) of the granule is in X. However, there is some uncertainty, the grade does not chooses how the equivalence class should be halved and which half the grade is referring to.

In rough set theory, we often face that: Objects belonging to the same equivalence class are not distinguishable. So U is not a classical set, but a multi-set or a bag [21]; U_C is a bag of multiple copies of one

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element. So the uncertainty mentioned in last paragraph disappears.

The main theme of this paper is to explore the phenomena rough set theory of the total space corresponds to a fuzzy set theory of the base space (quotient set) and vice versa (may not be uniquely).

1.3. Fuzzy Sets and Main Results

Note that a real number between 0 and 1 inclusive is not necessary a probability; see appendix. Similarly, a unit-interval valued function alone may not be regarded as a membership function of a fuzzy set. Formally,

A fuzzy set is an element of a collection of fuzzy sets (W-sofsets in the terminology of [14]), in which there are well defined set operations (for example, s- and t-norm) among themselves.

Somewhat surprisingly, from this prospect, we have the following conclusions:

The grades in the classical fuzzy set (W-sofsets) theories with s-and t-norm can not be interpreted as probabilities.

Granular fuzzy set theories are the theories, in which a grade can be interpreted as a probability (possibility, belief function, or others).

2. Rough Sets Fundamentals

Roughly, rough set theory has two major components, approximations and representations. The approximation theory is rooted in (pre-) topological spaces [15,18,23]. It is a natural notion. Similar ideas have been studied in approximate retrieval under various names, such as, goal query, query relaxation, rules extractions and others [1,2,16,17,24]. Some of these results are summarized in [7, 8, 9].

Loosely speaking the representation theory is the theory of relation instances (snapshots of relational databases), called information tables, information systems or knowledge representation systems. Its theory is useful in data mining and knowledge discovery. Recently, we have extended such representations to crisp/fuzzy binary relations, fuzzy coverings and fuzzy neighborhood systems [7].

Let U be the universe of discourse and R be an equivalence relation that partitions the universe into disjoint equivalence classes (granules). By abuse of notation, we will use R to denote the partition too. Let u $\in$ U and [u] be the equivalence class containing u. Note that [u] plays two roles; it is a subset of U, also an element of quotient set, U/R. To avoid confusing, we will use [u] to denote the role as a subset of U, and cname([u]) as an element of U/R. and said it is the canonical name [6].

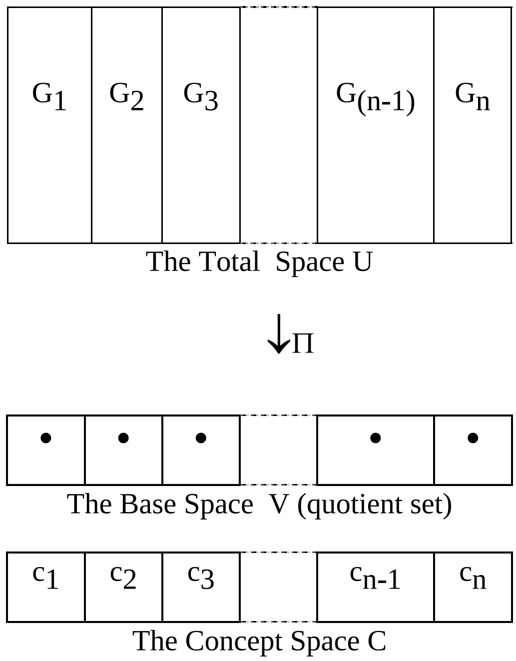

Figure 1.Granular structure for rough set theory

Let us collect some notations

U: the total space
R: the equivalence relation or the corresponding partition
V=U/R: the quotient set
[u]: the equivalence class (granule) containing u; it is a subset of U.
cname([u]): canonical name of [u]; it is an element of quotient set.
name([u]): the meaningful name of the equivalence class [u].
C ($\cong$ V): the base space that consists of all the meaningful names of equivalence classes; it was called concept space.
Π: the projection of U to C which is the composition of the natural projection and naming maps, that is,

$$u \to [u] \to c = \text{name}([u]). \quad (1)$$

3. "Rough Fuzzy" Sets – Finite Granular Fuzzy Sets

We quote the “rough fuzzy sets,” to distinguish ourselves from, the same term used in [3]. Some of these results has been reported in conferences [7, 8, 9, 12, 13, 15].

3.1. The Granular Structure

The most important structure the rough set theory gives us is the granular structure, (U, Π, C), the total space U, the equivalence relation, and the concept space C [8]. In this paper, we use the projection Π: u → name ([u]), to represent the equivalence relation.

In some branch of geometry, such a structure is called a fibre space. In Figure 1, the bold dots, $c_1, c_2, c_3, \ldots, c_n$, represent the member of base space. The "small vertical" rectangles, $G_1, G_2, G_3, \ldots, G_n$, denote the equivalence classes; each can be expressed by

$$G_i = \Pi^{-1}(c_j), \tag{2}$$

or equivalently

$$c_j = \mathrm{NAME}(G_j). \tag{3}$$

The goal of this paper is to build a fuzzy set theory on the base space C in terms of crisp set theory on the total space U.

3.2.Rough and Granular Membership Functions

Let X be a subset of the total space. Pawlak and Skworon [19, 20] defined rough membership functions and investigated them in terms of information tables. A rough membership function is defined by: (|•| is the cardinal number)

$$F_X(x) = \frac{|X \cap [x]|}{|[x]|}. \tag{4}$$

All members of the same equivalence class have the same grade. So naturally, it induces a grade on the base space (quotient set)

$$F_X(\mathrm{cname}([x])) = \frac{|X \cap [x]|}{|[x]|}. \tag{5}$$

The membership function on the base space will be called *granular membership function*. Classical set operations induces fuzzy set theoretical operations by:

$$F_X \cap F_Y(\mathrm{cname}([x])) = \frac{|X \cap Y \cap [x]|}{|[x]|} \tag{6}$$

$$F_X \cup F_Y(\mathrm{cname}([x])) = \frac{|X \cup Y \cup [x]|}{|[x]|} \tag{7}$$

These membership functions take rational numbers only; we will consider the general case in the next section. First, we need some terminology.

Definitions

1. The subset X ⊆ U is a crisp representation of the membership functions F_X on V; X in general is not unique; we have called it realization [11,12]. Conversely,
2. The membership function F_X on C is the (unique) *fuzzy representation* of X; note that the fuzzy set is on V and X is in U.

3.3. Crisp Set Representations of Fuzzy Sets

So far we have shown that each subset of the total space can be interpreted as a fuzzy set on the base space. The natural question is: Can every fuzzy set be so interpreted? In this section, we will give a positive answer to this question in finite universes.

We will illustrate the idea by an example. For simplicity, let us assume n = 9. Let F be a membership function defined on C as follows:

$F(c_1) = 1/6$, $F(c_2) = 1/3$, $F(c_3) = 1/2$, $F(c_4) = 2/3$,
$F(c_5) = 5/6$, $F(c_6) = 2/3$, $F(c_7) = 1/2$, $F(c_8) = 1/3$,
$F(c_9) = 1/6$.

We need to construct a granular structure: Consider a finite set S of 6 elements, and take the Cartesian product, U= S × C, as the total space. The map (s, c) → c as the projection Π; the granular structure is illustrated in Figure 2. Each square represents an element in U. All the squares in a vertical column, G_j, maps to the bold dot c_j under it.

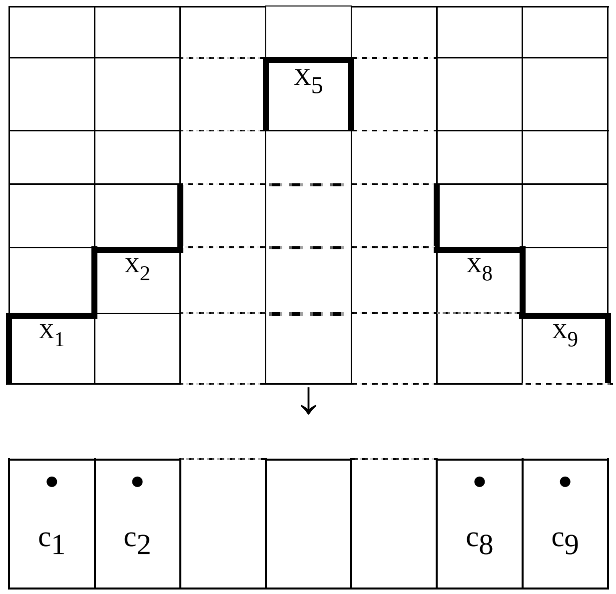

Figure 2. The crisp representation X in U for a membership function F defined on C

Next, we construct the crisp representation, X, that is, a subset satisfies the following:

$F(c_1) = |X \cap G_1| / |G_1|$, $F(c_2) = |X \cap G_2| / |G_2|$,
$F(c_3) = |X \cap G_3| / |G_3|$, $F(c_4) = |X \cap G_4| / |G_4|$,
$F(c_5) = |X \cap G_5| / |G_5|$, $F(c_6) = |X \cap G_6| / |G_6|$,
$F(c_7) = |X \cap G_7| / |G_7|$, $F(c_8) = |X \cap G_8| / |G_8|$,
$F(c_9) = |X \cap G_9| / |G_9|$.

Let X_j denote the subset of G_j that consists of the initial j squares (from the bottom square to X_j); see well ordering in the next paragraph. Note that $|X_j| = j$, for $j \le 5$ and $|X_{5+j}| = 5-j$ for $j < 5$. From Figure 2, it is easy to verify that

$$|X_j| / | G_j | = j/6 = F(c_j), \text{ if } j \le 5 \quad (8)$$

$$|X_j| / | G_j | = (5-j) /6 = F(c_j), \text{ if } j > 5 \quad (9)$$

Let $X = \cup^{9}_{j=1} X_j$. Then, X is the desired *crisp representation* of F.

3.3.1. Canonical Representations

Of course, the crisp representations may not be unique. In the following theorem, we well-order each equivalence class (well order principle of set theory), and consider only the subsets of initial segments; such a representation is called a *canonical representation*.

Theorem Let C be a finite crisp set and F_X be a rational valued membership function defined on C. Then, there is a granular structure (U, Π, C) and $X \subseteq U$ such that

$$F(c_j) = |X \cap G_j| / | G_j |. \quad (10)$$

Proof: Guided by the example, the construction of such an X is rather straightforward. By assumption, the membership is a fraction of two integers,

$$F(c_j) = p_j / q_j . \quad (11)$$

Let f = l.c.m $(q_1, q_2, \ldots, q_n)$ be the least common multiple. Consider the set S ={1, 2, . . . , f}, and set the Cartesian product $S \times C$ to be U. Then U, C and the projection to $S \times C \rightarrow C$ is the desirable granular structure. We take the initial segment of S for $X_j = \{1,2, \ldots, F(c_j)\}$, and consider their union X, which is the desired representation.Note that such an F(-) is not a probability measure, because in each equivalence class, we only consider the initial segments; they do not form a σ-algebra**.** The rationality condition can be dropped, if instead of counting we use probability measure, which will be covered in next section.

If $\forall j$, G_j is a bag, that is, all elements are identical [21, Appendix], then the crisp representation is unique.

Q.E.D.

Theorem If all elements in each equivalence class are identical, namely it is a bag consisting of the same element, then granular fuzzy operations are the same as MAX and MIN.

Proof: Let X and X' be two representations, that is

$$F(c_j) = |X \cap G_j| / | G_j | = |X' \cap G_j| / | G_j |. \quad (12)$$

By the properties of bags,

$$|X \cap G_j| = |X' \cap G_j| \Rightarrow X \cap G_j = X' \cap G_j, \forall j,$$

$$\Rightarrow X = X'. \quad (13)$$

Q.E.D.

4. Granular Fuzzy Sets, Probability and Measurements

A fuzzy set is defined solely by a membership function, in which the notion of grades is *not* given any mathematical meaning [26] (pp.12). They have been interpreted intuitively only. If we take such a view to an extreme, then fuzzy set theories are merely some theories of functions, and hence is a sub-discipline of mathematical analysis. We believe fuzzy sets are more than that. From rough set point of view, which is treated in last section, the grade of a membership represents the counting of a partial membership of an equivalence class; this covers only the rational valued memberships.

In this section, we extend the counting to probability and other forms of measurements. Let us re-iterate that from a pure mathematical point of view, an abstract measure (with total measure =1) is a probability. In other words, a normalized measure is a probability and vice versa. The notions of "partial truth" and "partial certainty" are the same from the mathematical operational point of view.

4.1. The Context of Granular Fuzzy Sets

In rough set theory, the universe of discourse is the pair, the total space and the equivalence relation. In granular fuzzy set theory, the universe of discourse is a granular structure. The rational is that in the real wold applications of fuzzy theory, there are always *some reference* structures for the grades. For example, the degree of redness, one either scientifically refers to wave length or intuitively the intensity of the feelings. If there is no *natural* granular structure, we can mathematically construct a product granular structure; see the following proof the theorem.

4.1.1. Probability

We will add probability measures to the granular structure, that is, a triple (U, Π, C), where each equivalence class, $U_c = \Pi^{-1}(c_j)$, is a probability space; its probability is denoted by μ_c.

4.1.2. General Measurements

Next, we consider more general case. We assume there is a general measurement defined on the given granular structure, namely, a real valued set function

$$\alpha_c : U_c \to R \quad (14)$$

is given. The family $\alpha = \{\alpha_c : c \in C\}$ is called a total general measurement on U. We say a total general measurement α is a counting measure, a probability, a possibility, or a belief function, iff $\forall c$, α_c: $U_c \to R$ is respectively the same. Mathematically, α is a family of measurements parameterized by the base space.

4.2. Granular Fuzzy Sets in a Given Context

Let X be a subset (with conditions specified below) of the total space; note that the total space is partitioned into measurable subspaces. So we assume X is a measurable set in each equivalence class, and define a function whose value at c to be

$$F_X(c) = \mu_c(X \cap U_c). \quad (15)$$

This F_X is a *granular membership function*. For general measurements, the definition is similar; we will skip them in this paper.

It is clear how a crisp set in the total space defines a fuzzy set on the base space. However, the converse, in general, is not clear:

1. for a given probability F(c), it is unclear there is a measurable set $X_c \subseteq U_c$ such that $\mu_c(X_c) = F(c)$.
2. In general, we can not well order equivalence class, U_c, in such a way that initial segments are measurable sets, So in general, there is no canonical representation for a given granular structure.

However, we have the following partial results.

Theorem Let C a crisp set, not necessarily finite. Let F be a membership function defined on C. Then there is a product granular structure (U, Π, C) such that

1. (U_c, μ_c) is a probability space for every $c \in C$, where $U_c = \Pi^{-1}(c)$.
2. There is a set $X \subseteq U$ such that

$$F(c) = \mu_c(X \cap \Pi^{-1}(c)) \quad (16)$$

Proof: The easiest U is the direct product of the unit interval and C. In this case $U_c = \Pi^{-1}(c)$ is a copy of the unit interval and the probability on U_c is the Lebesque measure. To find X, for each c, we take X_c to be a sub-interval, that is,

$$X_c = [0,\ F(c)]. \quad (17)$$

Then set

$$X = \cup_c [0,\ F(c)]. \quad (18)$$

This is a canonical representation; since all subsets X_c are the initial segments.

Q.E.D.

4.3. Granular Fuzzy Set Operations

The operations of granular membership functions are defined as follows:

(1) $F_X(c) = \mu_c(X \cap U_c)$; (19)

(2) $F_{(X \oplus Y)}(c) = \mu_c[(X \oplus Y) \cap U_c]$; (20)

(3) $F_{(X \otimes Y)}(c) = \mu_c[(X \otimes Y) \cap U_c]$. (21)

where $\oplus$ and $\otimes$ are the abstract operations of "union" and "intersection."

If we replace all μ_c by α_c, we have the general granular fuzzy sets (we assume α_c is normalized).

In next section, we will show most of classical operations can be expressed by granular fuzzy operations. First, we make a simple observation.

Proposition Granular fuzzy operations are not truth functional.

Proof: From Figure 3, it is easy to see that

$$F_{(X \cup Y)} = F_X + F_Y; \quad (22)$$

$$F_{(X \cap Y)} = 0. \quad (23)$$

and the grade of the intersection, $F_{(X \cap Y)}$, can not be computed from the grades of two factors, F_X and F_Y.

Q.E.D.

Corollary The grades in classical fuzzy sets (with s- and t-norms) can not be probabilities.

This is immediate from the fact that s- and t-norms are truth functional.

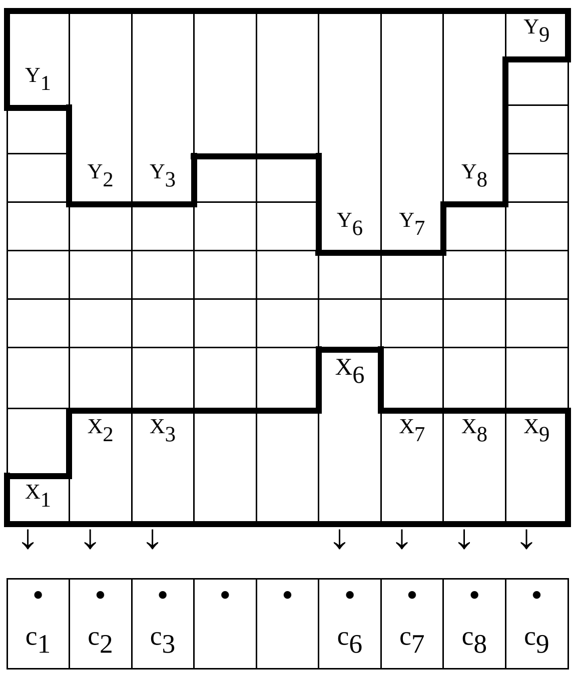

Figure 3. Granular fuzzy sets, X and Y in Example 1

Example 1

1.1. Let X be a crisp representation of F_X indicated in Figure 3. Each X_j denotes all the boxes **below** it.

1.2. Let Y be a crisp representation of F_Y. Each Y_j denotes all the boxes **above** it.

Example 2

2.1. Let X be a crisp representation of F_X indicated in the figure below. Each X_j denotes all the boxes **below** it.

2.2. Let Y be a crisp representation of F_Y. Each Y_j denotes all the boxes **below** it.

From Figure 4 and 5, it is easy to get

$$F_{(X \cup Y)} = \text{Max}\,(F_X, F_Y) \quad (24)$$

$$F_{(X \cap Y)} = \text{Min}\,(F_X, F_Y) \quad (25)$$

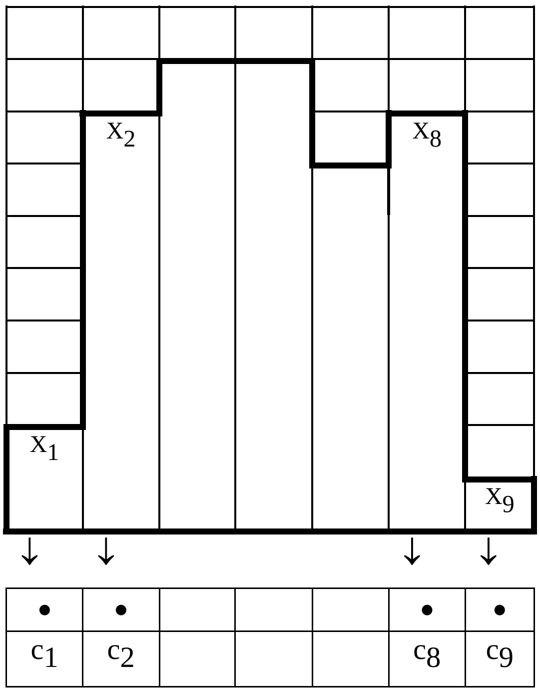

Figure 4. Granular fuzzy set X in Example 2

For this example, the membership functions $F_{(X \cup Y)}$, $F_{(X \cap Y)}$ are the same as usual Max and Min. The reason is that both X and Y take canonical crisp representations. In this particular example, both X and Y take *the squares "from bottom up*" (the initial segments). However, in Example 1, X and Y are taken from different areas. These two examples clearly show that the Max and Min only work for canonical crisp representations.

5. Classical t- and s- Operations

In this section, we will show that classical fuzzy sets with s-and-t-norm can be represented by granular fuzzy sets.

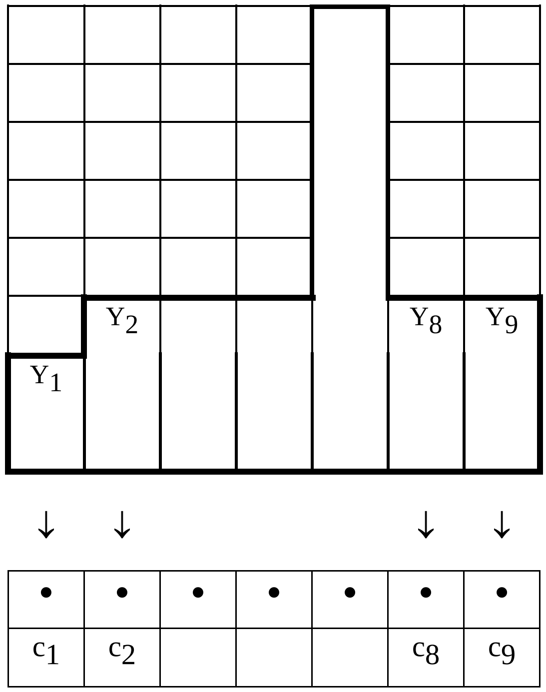

Figure 5. Granular fuzzy set Y in Example 2

The essential idea is that each number r in the unit interval will be represented by the interval [0, r]. In this way, the traditional grade can be interpreted as the Lesbegue measure (a probability) of an interval. So the grade is a measure of a measurable set X_c, an interval.

The following proof has been reported in a conference [10]. Let U = C × [0, 1] be the total space, and $\Pi : U \to C$ be the projection. For each $U_c = c \times [0, 1]$, we set the probability

$$\mu(c \times S) = \mu(S), \quad (26)$$

where $\mu(S)$ be the Lebesgue measure (probability) of S $\subseteq [0,1]$. Next, we will construct a crisp representation X for the given membership function F as follows: At each point $c \in C$, we set

$$I_c = [0, y_c) \text{ such that } F(c) = y_c, \quad (27)$$

$$X_c = c \times I_c : c \in C, \text{ and} \quad (28)$$

$$X = \cup_c X_c \quad (29)$$

Next we introduce "intersection" ⊗ and "union" ⊕. Consider two sets X_i, i=1, 2. At each point c we have

$$X_{ic} = c \times [0, y_{ic}), \ y_{ic} \in [0, 1], i=1, 2. \quad (30)$$

Next, we define the operations:

$$X_{1c} \otimes X_{2c} \equiv (c \times [0, y_{1c})) \otimes (c \times [0, y_{2c})) = c \times [0, y_{3c}), \quad (31)$$

where $y_{3c} = t(y_{1c}, y_{2c})$ is the usual t-norm.

$$X_{1c} \oplus X_{2c} \equiv (c \times [0, y_{1c}))\oplus (c \times [0, y_{2c}))$$
$$= c \times [0, y'_{3c}), \quad (32)$$

where $y'_{3c} = s(y_{1c}, y_{2c})$ is the usual s-norm. We write

$$X_1 = \cup_c X_{1c} \quad (33)$$

and

$$X_2 = \cup_c X_{2c}. \quad (34)$$

Globally, we define

$$X_1 \otimes X_2 \equiv \cup_c X_{1c} \otimes X_{2c} \quad (35)$$

and

$$X_1 \oplus X_2 \equiv \cup_c X_{1c} \oplus X_{2c} \quad (36)$$

Or equivalently

$$(X_1 \otimes X_2)\cap U_c = X_{1c} \otimes X_{2c} \quad (37)$$

and

$$(X_1 \oplus X_2)\cap U_c = X_{1c} \oplus X_{2c} \quad (38)$$

With such definitions, we immediately have the following granular fuzzy set operations

$$F_1 \otimes F_2 (c) = \mu_c(X_1 \otimes X_2 \cap U_c)$$
$$= \mu(c \times [0, y_{3c})) = t(G_1, G_2) \quad (39)$$

$$F_1 \oplus F_2 (c) = \mu_c(X_1 \oplus X_2 \cap U_c)$$
$$= \mu(c \times [0, y'_{3c}))= s(G_1, G_2) \quad (40)$$

These discussions conclude the following:

Theorem Given a family $\{F_i\}$ of fuzzy sets which are closed under s-and t- norms, then there is a family of granular fuzzy sets $\{(F_i, X_i)\}$ which are closed under granular fuzzy set operations, $\oplus$ and $\otimes$, such that the mapping $(F_i, X_i) \rightarrow F_i$ is an isomorphism using respective fuzzy operations.

This is an easy theorem, it merely represents a real value r by the interval [0, r) of that length. However, it does have some meaningful implications:

1. It is clear there are many crisp representations for a given membership function.
2. An arbitrary choice of crisp representations does *not* induce the classical s- and t- norms. It implies that classical s- and t- norms are based on some specific contexts.

We would like to stress that in many fuzzy applications, the context, a granular structure (U, Π, C), is given *naturally* and granular fuzzy sets are *readily applicable*.

6. Conclusion

Both fuzzy and rough set theories have been proved to be effective in handling uncertainty. In this paper, we explore one aspect of their interactions. The central notions of rough set theory are the total space and an equivalence relation that partitions the total space into mutually equivalence classes. Objects belonging to the same equivalence class is indiscernible. By considering the universe, called total space, the quotient set, called base space, the natural projection, we find that we have a nice framework, called a granular structure. It bridges the two theories rather nicely. Crisp sets (concepts) of total space correspond to fuzzy sets (fuzzy concepts) of base space; however, the converse may not be unique. If the equivalence relation is truly indiscernible, then it is unique. From the bridge we derive naturally the notion of granular fuzzy sets. We believe that granular view (rough set) of fuzzy sets may give fuzzy set theory a new direction and foundation, we believe many applications will be derived.

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Appendix-Probability

As we have mentioned several times, a common misconception is any real value in the unit interval can be interpreted as a probability. This is far from the truth. Mathematically a probability theory is a measure theory in which the total measure of the universe is one. Roughly, probabilities are a collection of numbers (measures of events) that satisfies certain computational rules, which are explained in below.

Let V be the universe, we will be interested in the following notions.

1. An algebra (or Boolean algebra) of sets is a non-empty class R of sets $\subseteq$ V such that
   (a) if $E \in R$ and $F \in R$, then $E \cup F \in R$
   (b) if $E \in R$, then E'
2. A σ-Ring (or σ-Boolean Ring) of sets is a non-empty class S of sets such that
   (a) if $E \in S$ and $F \in S$, then $E - F \in S$
   (b) if $E_i \in S$ then $\cup_i \{E_i \mid I = 1, 2, \ldots\} \in S$

   A σ-algebra is σ-ring containing V.
3. A measure is an *extended real valued*, *non-negative*, and *countably additive* set function μ, defined on a σ-ring, and such that μ(0)=0.
4. A probability is a set function defined on a σ-algebra, in which
   4.1. it is nonnegative extended real valued

$$0 \le \mu(X) \le 1 \tag{41}$$

   4.2. countably additive:

   For $E = \cup_n^\infty E_n$ and $E_i \cap E_j = \emptyset$, i, j = 1, 2, ...

$$\mu(E) \equiv \Sigma_n^\infty \mu(E_n). \tag{42}$$

   4.3. μ(X)=1 and μ(0)=0. (43)

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