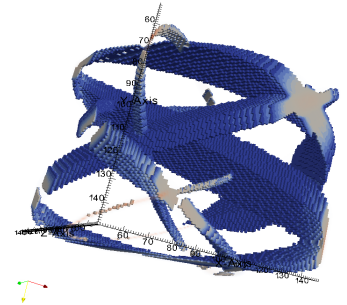
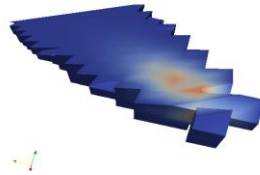
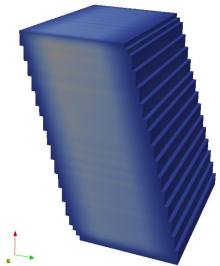


Linear Dynamics of an Elastic Beam and Plate Under Moving Loads with Uncertain Parameters



Andrzej Pownuk

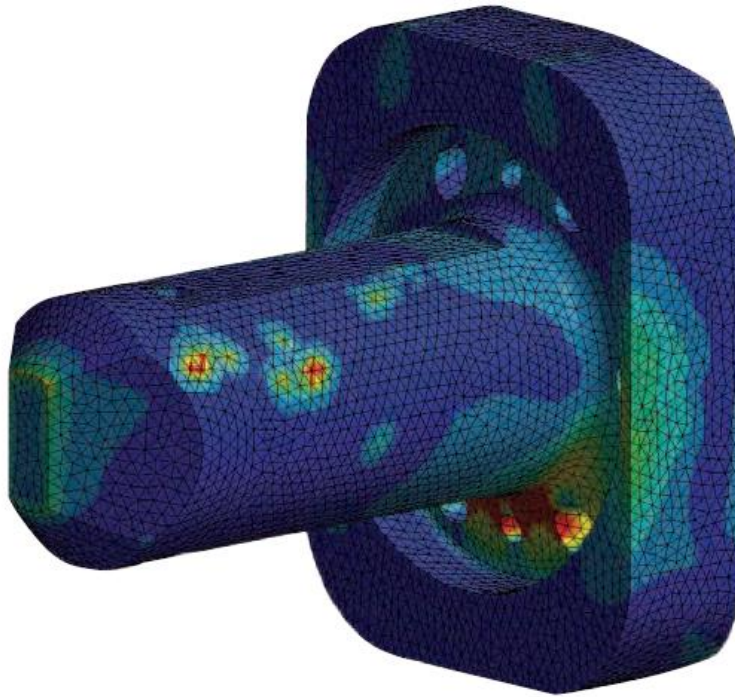
The University of Texas at El Paso

<http://andrzej.pownuk.com>

Outline of the presentation

- Equations with the uncertain parameters and their applications
- New approach for the solution of the equations with the interval parameters
- Generalizations and conclusions

Mathematical model of a machine



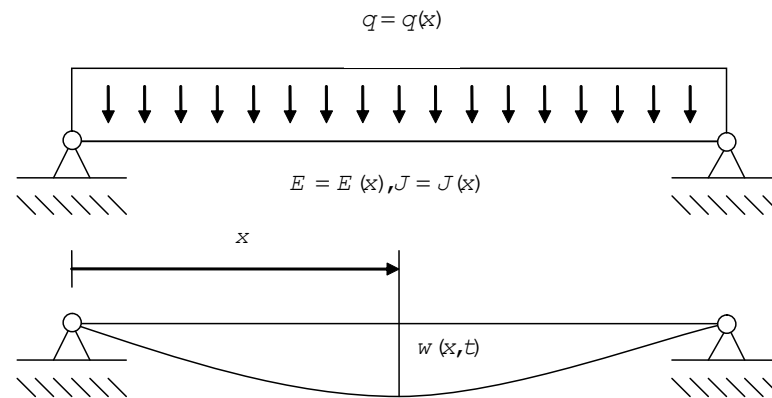
$$\left\{ \begin{array}{l} \sum_{j=1}^3 \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \\ \sigma_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 C^{ijkl} \varepsilon_{kl} \\ \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ u_i = u_i^*, x \in \partial V_u \\ \sum_{j=1}^3 \sigma_{ij} n_j = t_i^*, x \in \partial V_\sigma \\ u_i = u_i(x, 0), x \in V \end{array} \right.$$

Such simulations are possible since early 1970s

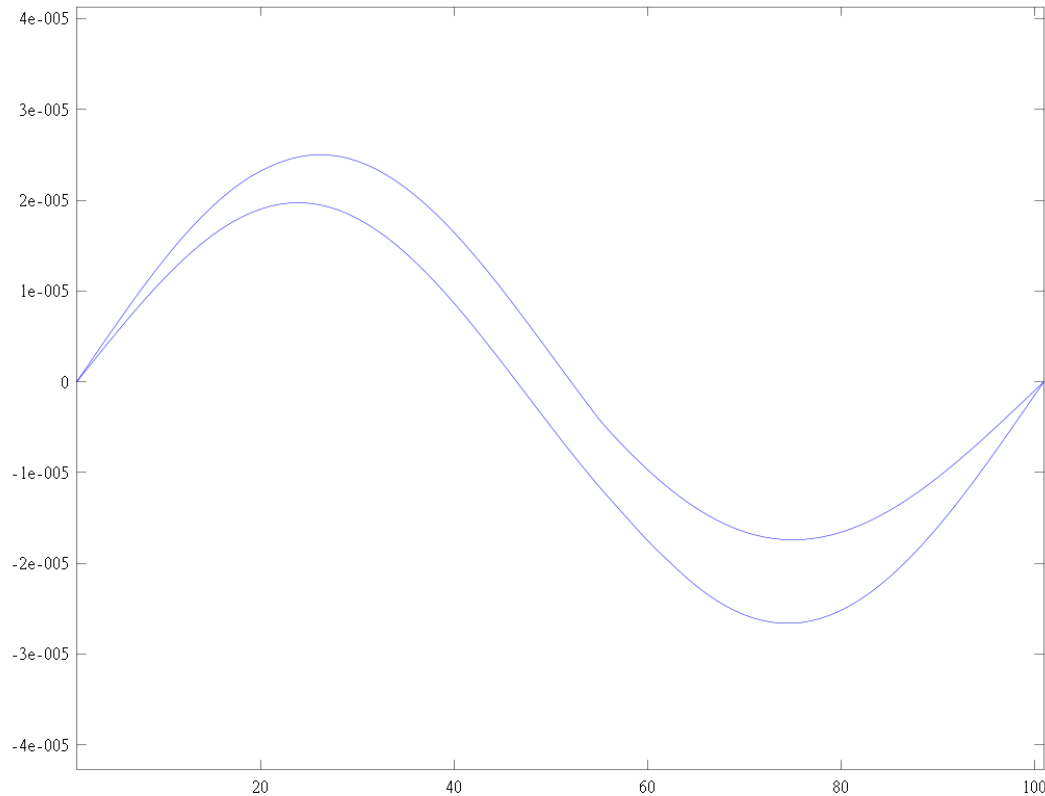
O.C. Zienkiewicz, Ivo M. Babuška, P.G. Ciarlet ...

- Beam model with the interval parameters

$$\left\{ \begin{array}{l} E J \frac{\partial^4 w}{\partial x^4} = q - \rho A \frac{\partial^2 w}{\partial t^2} \\ w(0, t) = 0 \\ w(L, t) = 0 \\ \frac{\partial^2 w(0, t)}{\partial x^2} = 0 \\ \frac{\partial^2 w(L, t)}{\partial x^2} = 0 \\ w(x, 0) = w_0(x) \\ v(x, 0) = \frac{\partial w(x, 0)}{\partial t} = v_0(x) \\ E \in \mathbf{E}, q \in \mathbf{q}, A \in \mathbf{A} \end{array} \right.$$

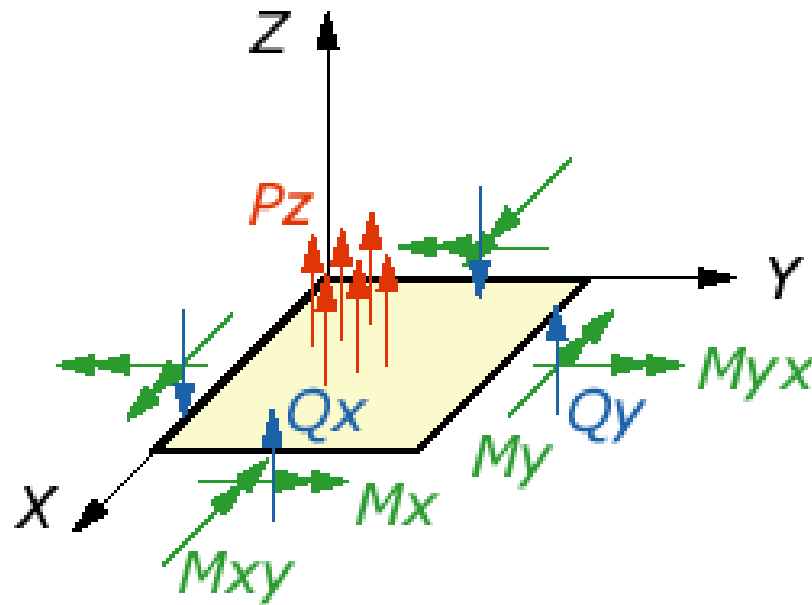


Interval displacements



- Plate with the interval parameters

$$\left\{ \begin{array}{l} D \left(\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} \right) = q - \rho h \frac{\partial^2 u}{\partial t^2} \\ u(0, y, t) = 0 \\ u(L, y, t) = 0 \\ u(x, 0, t) = 0 \\ u(x, L, t) = 0 \\ \frac{\partial^2 u}{\partial x^2}(0, y, t) = 0 \\ \frac{\partial^2 u}{\partial x^2}(L, y, t) = 0 \\ \frac{\partial^2 u}{\partial y^2}(x, 0, t) = 0 \\ \frac{\partial^2 u}{\partial y^2}(x, L, t) = 0 \\ u(x, y, 0) = u^*(x, y) \\ \frac{\partial u(x, y, 0)}{\partial t} = v^*(x, y) \\ E \in \mathbf{E}, q \in \mathbf{q}, h \in \mathbf{h} \end{array} \right.$$



Mathematical models

physical problem



$$\begin{aligned}\sigma_{ij,j} + F_i &= 0 \text{ in } \Omega, \\ \epsilon_{ij} &= \frac{1}{2}(u_{j,i} + u_{i,j}), \\ \sigma_{ij} &= C_{ijkl} \epsilon_{kl}.\end{aligned}$$

mathematical models

cheap

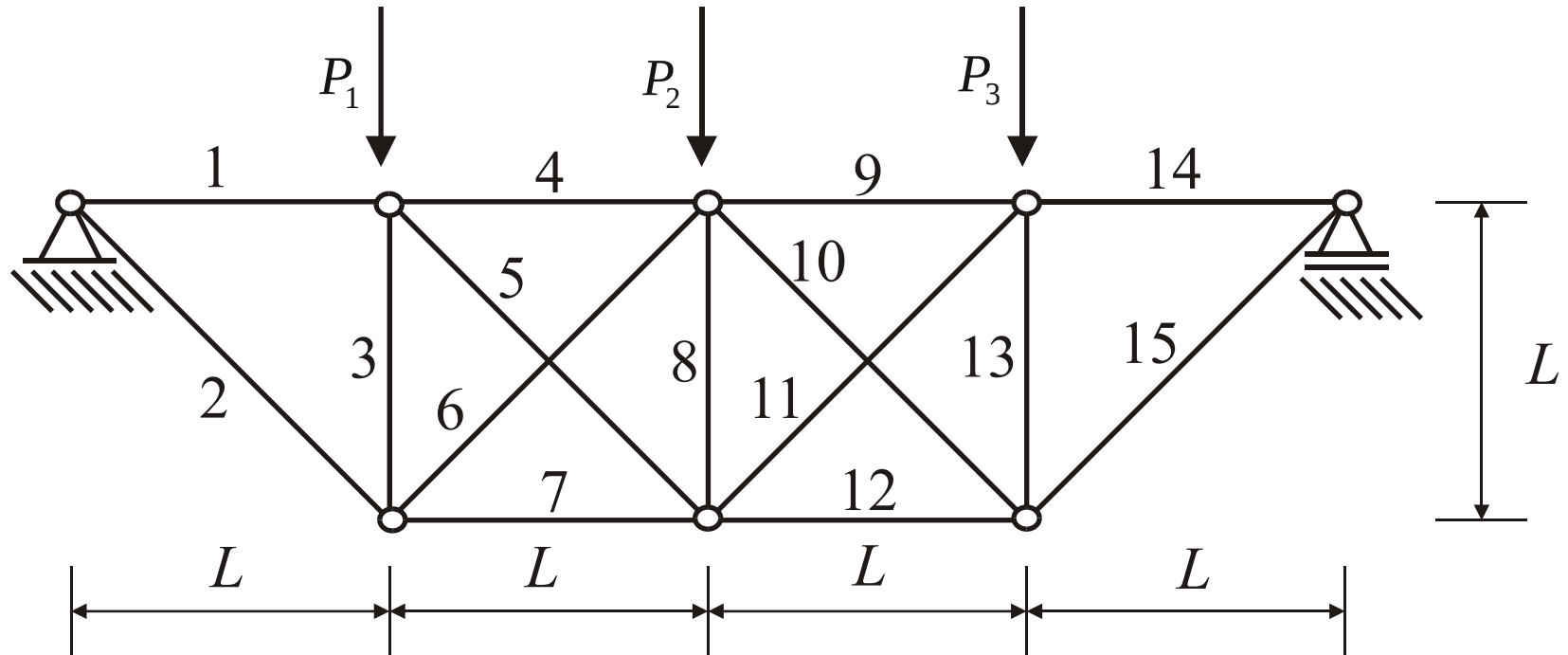
expensive



experiments

experimental results $\approx$ predictions

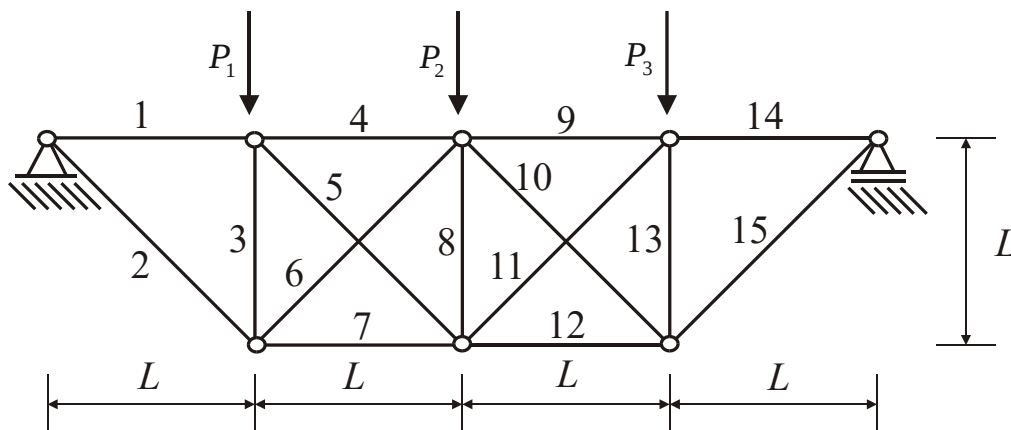
Truss structure with uncertain forces



Perturbated forces

$$P = P_0 \pm \Delta P \quad 5\% \text{ uncertainty}$$

No	1	2	3	4	5	6	7	8
ERROR %	10	9,998586	10,00184	10,00126	60,18381	11,67825	9,998955	31,8762
No	9	10	11	12	13	14	15	
ERROR %	10,00126	11,67825	60,18381	9,998955	10,00184	10	9,998586	



Uncertainty

Problem with real parameters

$$2x = 4$$

$$x = \frac{4}{2} = 2$$

$$2 \in [1, 3]$$

$$4 \in [3, 5]$$

Problem with interval parameters

$$[1, 3]x = [3, 5]$$

$$x = ?$$

Algebraic Solution

$$[1, 2]x = [1, 4]$$

$$x = [1, 2]$$

because

$$[1, 2] \cdot [1, 2] = [1, 4]$$

United Solution Set

$$[1, 2]x = [1, 4]$$

$$\mathbf{x} = \frac{[1, 4]}{[1, 2]} = \left[\frac{1}{2}, 4 \right]$$

because

$$\mathbf{x} = \{x : ax = b, a \in [1, 2], b \in [1, 4]\}$$

Comparison of the solution sets

$$[1, 2]x = [1, 4]$$

$$\mathbf{x} = [1, 2] \quad \neq \quad \mathbf{x} = \frac{[1, 4]}{[1, 2]} = \left[\frac{1}{2}, 4 \right]$$

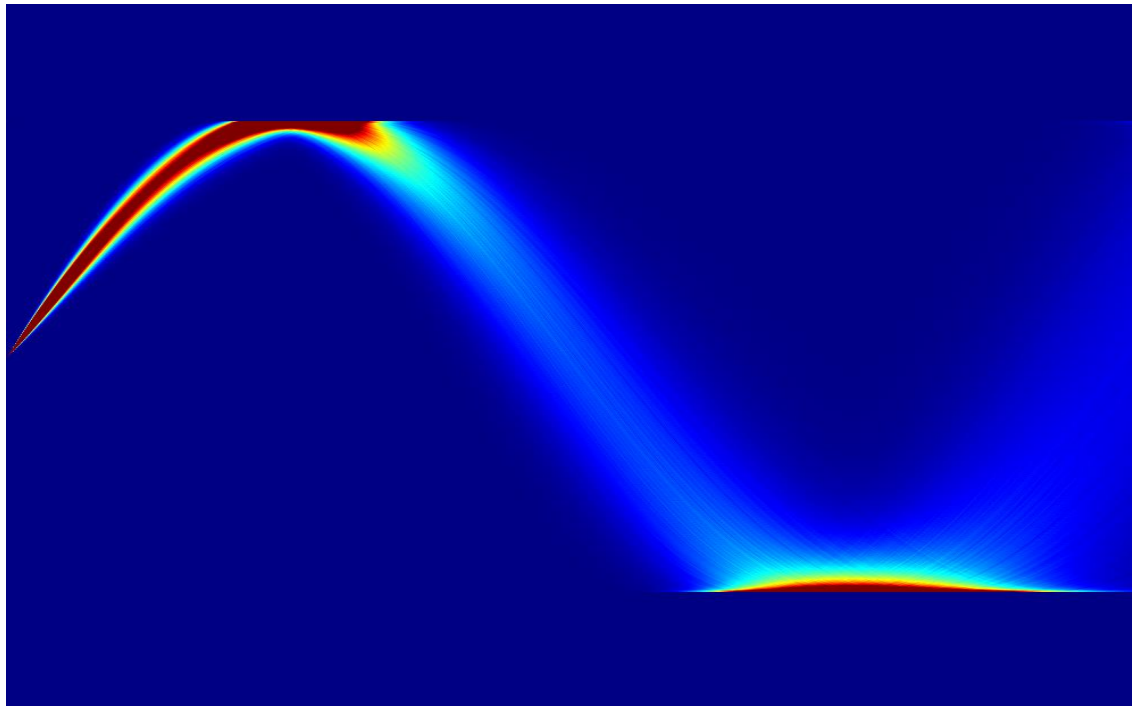
Algebraic Solution

United Solution Set

There are many ways how it is possible to extend equations with the real parameters into equations with the interval parameters.

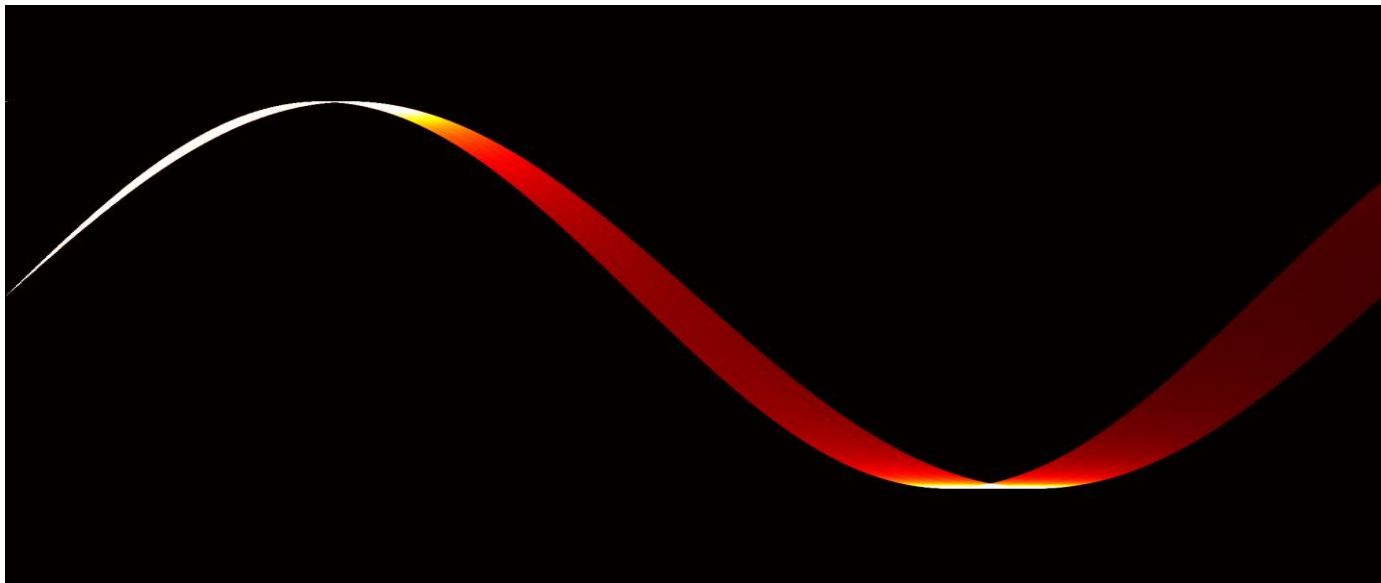
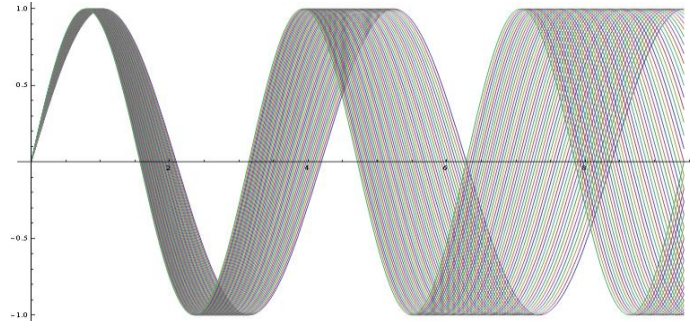
Stochastic differential equations

$$\begin{cases} y' = p \cos(pt) \\ y(0) = 0 \\ p \sim N(0,1) \end{cases}$$



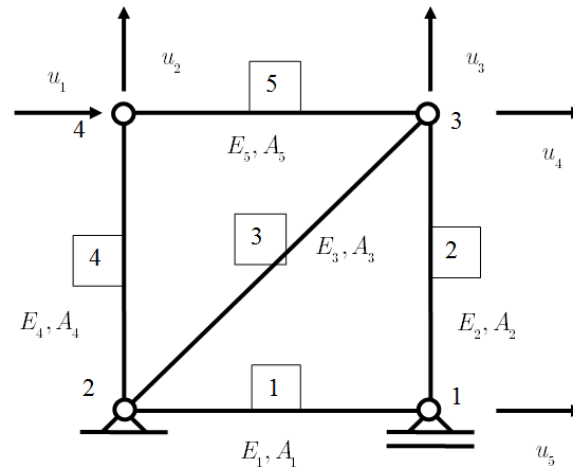
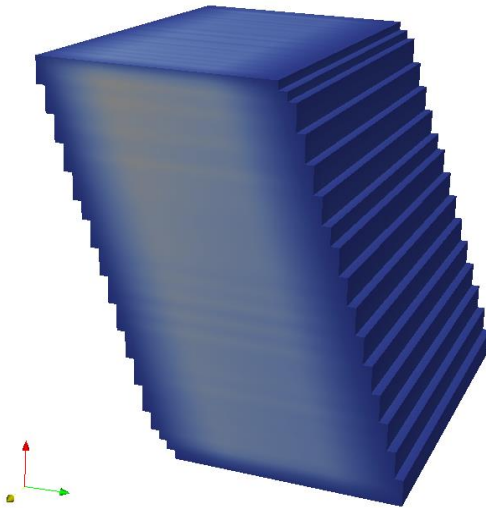
Interval equation

$$\begin{cases} y' = p \cos(pt) \\ y(0) = 0 \\ p \in [\underline{p}, \bar{p}] \end{cases}$$

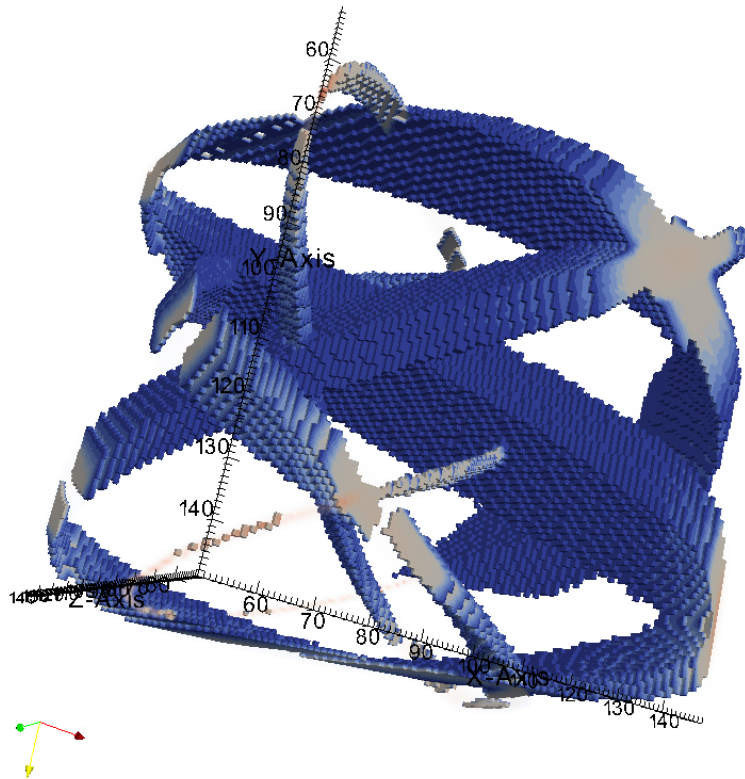


Solution set in 3D

$$u(\mathbf{p}) = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt{2}P_1}{A_3E_3} + \frac{P_1}{A_5E_5} + \frac{P_1 + P_3}{A_2E_2} \\ -\frac{P_2}{A_4E_4} \\ -\frac{P_1 + P_3}{A_2E_2} \end{bmatrix} : E_i \in [\underline{E}_i, \bar{E}_i], P_i \in [\underline{P}_i, \bar{P}_i] \right\}$$



Solution set in 3D



$$\begin{cases} \frac{dx}{dt} = pr \cos(pt) \\ \frac{dy}{dt} = -pr \sin(pt) \\ \frac{dz}{dt} = p \end{cases} \quad , p \in [\underline{p}, \bar{p}], r \in [\underline{r}, \bar{r}]$$

$$\begin{aligned} x(0) &= 0 \\ y(0) &= 1 \\ z(0) &= 0 \end{aligned}$$

Automatically generated test problems

Interval FEM Examples

List of examples

Truss examples

- [Truss 1x1, 1 force, 5 interval parameters](#)
- [Truss 1x1, 3 force, 13 interval parameters](#)
- [Truss mxn](#)
- [Truss benchmark 1 Compare Truss benchmarks](#)
- [Truss benchmark 2 Compare Truss benchmarks](#)
- [Truss benchmark 1a Compare Truss benchmarks](#)
- [Truss benchmark 2a Compare Truss benchmarks](#)

[Truss example \(similar to benchmark 2\)](#)
[Truss example \(similar to benchmark 2\)](#)
[Truss example \(page 185 from prof. Muhanna paper's\)](#)
[Truss example geometric uncertainty \(page 185 from prof. Muhanna paper's\)](#)

Frame examples

- [Frame \(2 elements\)](#)

2D examples

- [2D-Elasticity point load](#)
- [2D-Elasticity point load \(2 loads only\)](#)
- [2D-Elasticity point load \(2 loads only\), dependent case](#)
- [2D-Elasticity point load \(2 point support\)](#)
- [2D-Elasticity point load \(2 point support\) - uncertain geometry](#)
- [2D-Elasticity gravity force](#)
- [2D-Elasticity surface load](#)

Generate the script for the calculation of the 2D elasticity problem with interval parameters with point loads.

Analysis type
Combinatoric solution

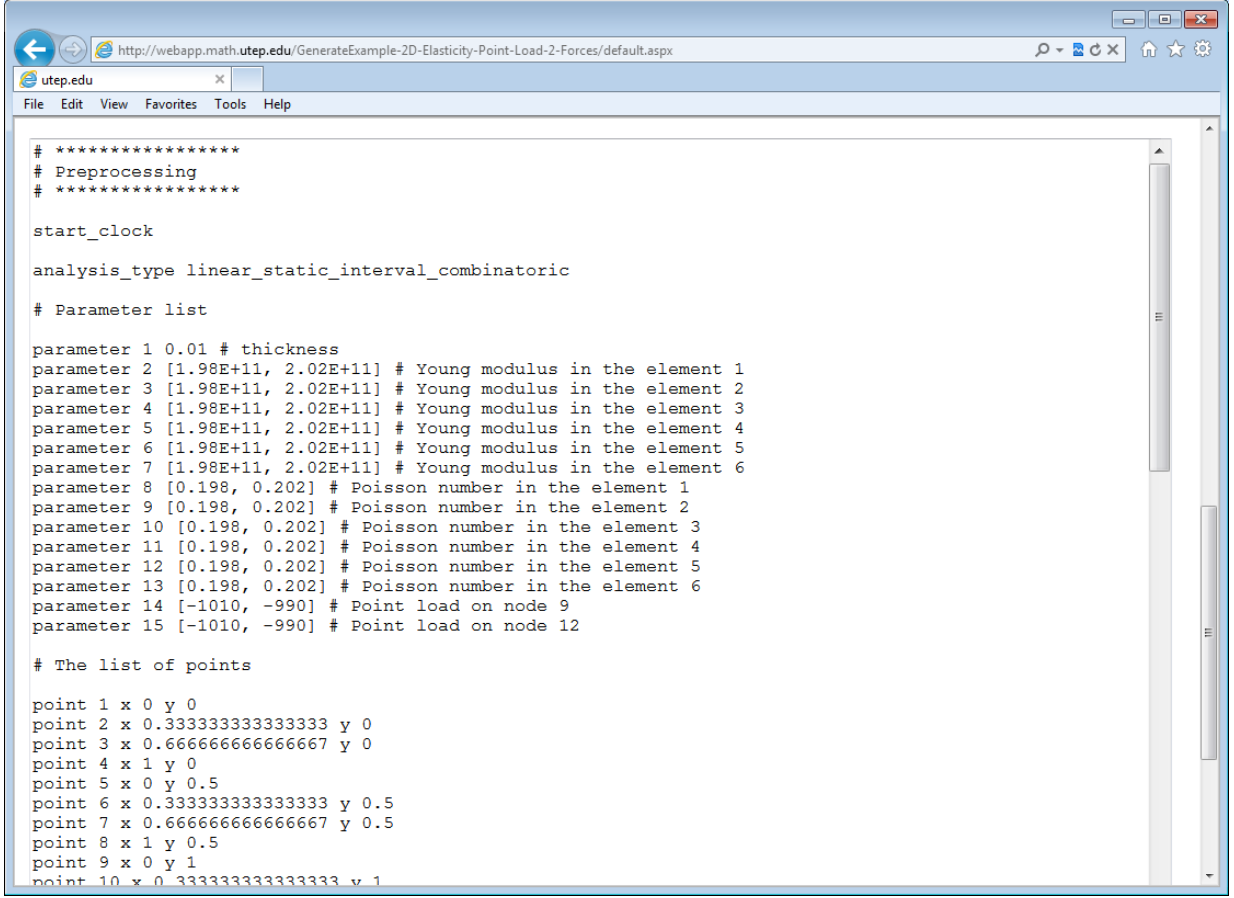
Name	Value	Uncertainty %
Young modulus	200E9	2
Poisson number	0.2	2
Point load	-1000	2
Width (L)	1	
Height (H)	1	
Number of elements in the x-direction (nx)	3	
Number of elements in the y-direction (ny)	2	
Thickness	0.01	

Generate the model

<http://webapp.math.utep.edu/Pages/IntervalFEMExamples.htm>

Automatically generated test problems

DSL
(Domain Specific Languages)



The screenshot shows a web browser window with the address bar displaying `http://webapp.math.utep.edu/GenerateExample-2D-Elasticity-Point-Load-2-Forces/default.aspx`. The browser's address bar also shows `utep.edu`. The browser's menu bar includes `File`, `Edit`, `View`, `Favorites`, `Tools`, and `Help`. The main content area displays a text-based DSL code for a 2D elasticity problem. The code is as follows:

```
# *****
# Preprocessing
# *****

start_clock

analysis_type linear_static_interval_combinatoric

# Parameter list

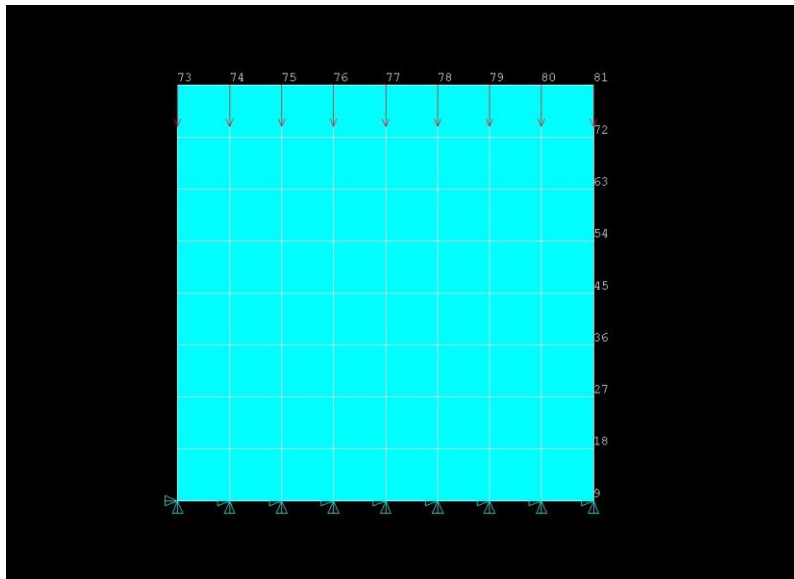
parameter 1 0.01 # thickness
parameter 2 [1.98E+11, 2.02E+11] # Young modulus in the element 1
parameter 3 [1.98E+11, 2.02E+11] # Young modulus in the element 2
parameter 4 [1.98E+11, 2.02E+11] # Young modulus in the element 3
parameter 5 [1.98E+11, 2.02E+11] # Young modulus in the element 4
parameter 6 [1.98E+11, 2.02E+11] # Young modulus in the element 5
parameter 7 [1.98E+11, 2.02E+11] # Young modulus in the element 6
parameter 8 [0.198, 0.202] # Poisson number in the element 1
parameter 9 [0.198, 0.202] # Poisson number in the element 2
parameter 10 [0.198, 0.202] # Poisson number in the element 3
parameter 11 [0.198, 0.202] # Poisson number in the element 4
parameter 12 [0.198, 0.202] # Poisson number in the element 5
parameter 13 [0.198, 0.202] # Poisson number in the element 6
parameter 14 [-1010, -990] # Point load on node 9
parameter 15 [-1010, -990] # Point load on node 12

# The list of points

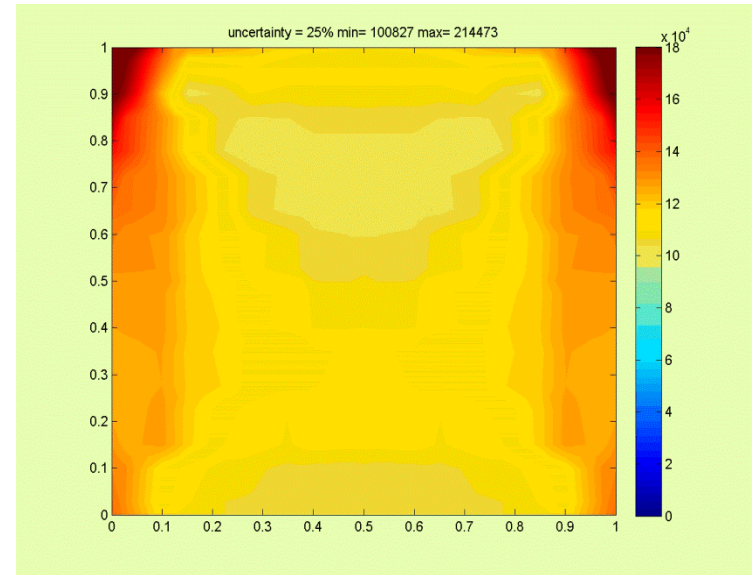
point 1 x 0 y 0
point 2 x 0.3333333333333333 y 0
point 3 x 0.6666666666666667 y 0
point 4 x 1 y 0
point 5 x 0 y 0.5
point 6 x 0.3333333333333333 y 0.5
point 7 x 0.6666666666666667 y 0.5
point 8 x 1 y 0.5
point 9 x 0 y 1
point 10 x 0.3333333333333333 y 1
```

2D elasticity problem with the interval parameters

Model



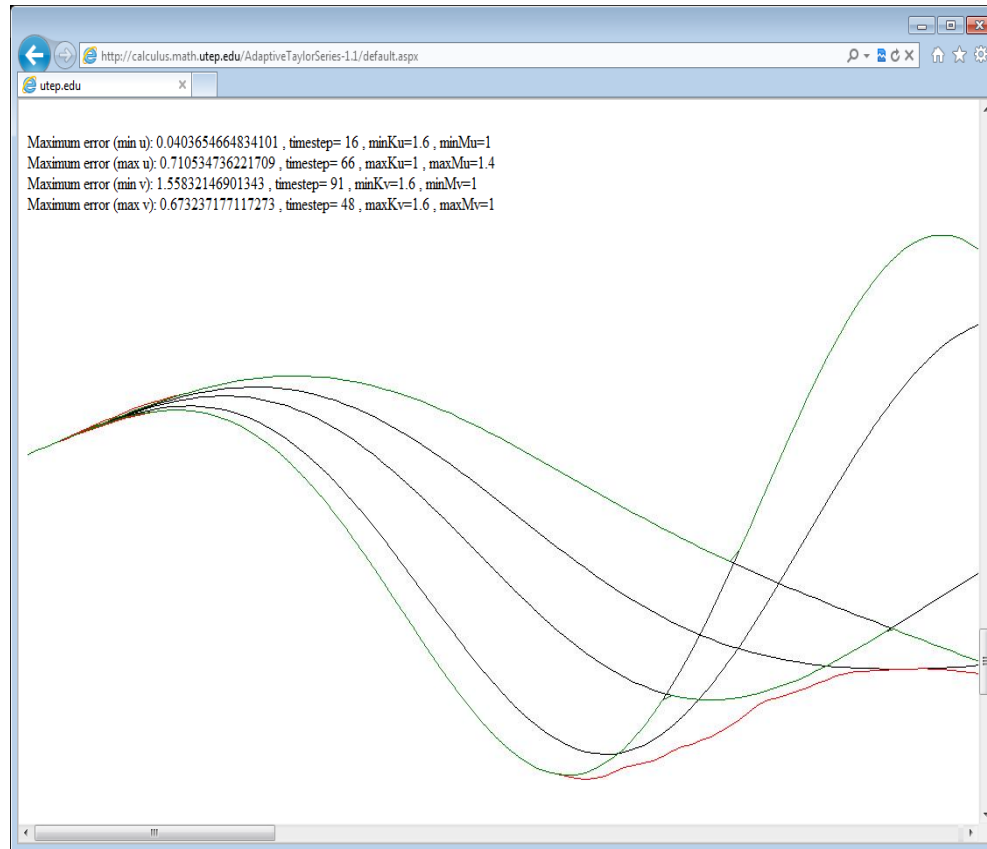
Solution



Mathematical model

$$\mu u_{i,jj} + (\mu + \lambda) u_{j,ij} + F_i = \rho \partial_{tt} u_i \quad \text{or} \quad \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}) + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}.$$

Adaptive Taylor series

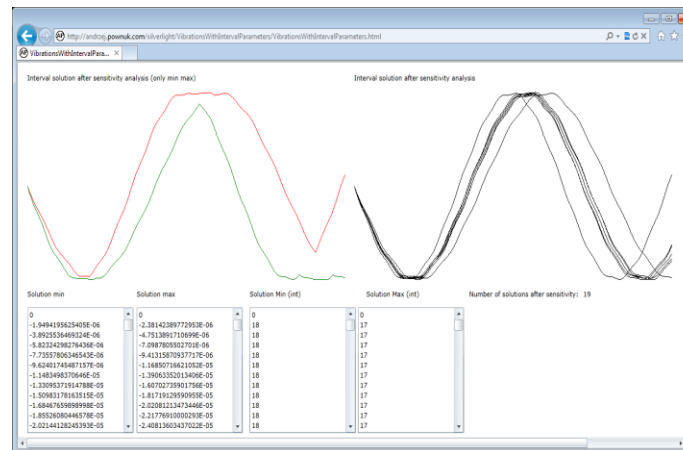


<http://webapp.math.utep.edu/AdaptiveTaylorSeries-1.1/>

Adaptive Taylor series

$E = 200E9$ $dE = 5\%$ $\min E = 190000000000$ $\max E = 210000000000$ $\rho = 7874$ [kg/m³]
 $A = 0.01$ $dA = 5\%$ $\min A = 0.0095$ $\max A = 0.0105$ $d\rho = 5\%$
 $I = 8.333E-6$ $dI = 5\%$ $\min I = 7.91633E-06$ $\max I = 8.74965E-06$ $\min \rho = 7480.3$
 $d = 0.001$ [s] $P = 1000$ [N] Time steps for load = 1 $\max \rho = 8267.7$
 $L = 10.0$ [m] $n = 1$ Total time when the load was applied = 0.001 [s]
 $L_n = 5$ [m]

Init calculations Number of interval parameters: 8
 List of nodes Number of timesteps: 400
 node 1, x = 0 Number of DOF = 9
 node 2, x = 5
 node 3, x = 10
 number of elements = 2
 Number of nodes = 3
 Dof in nodes
 node 0 0 1 2
 node 1 3 4 5
 node 2 6 7 8
 Nodes in elements
 element 0 0 1
 element 1 1 2
 DOF in elements
 element 0 0 1 2
 element 1 3 4 5
 element 1 6 7 8



<http://andrzej.pownuk.com/silverlight/VibrationsWithIntervalParameters/VibrationsWithIntervalParameters.html>

Tools which support my research

Epistemic uncertainty

H – set of horses

This is a horse.



$\in H$

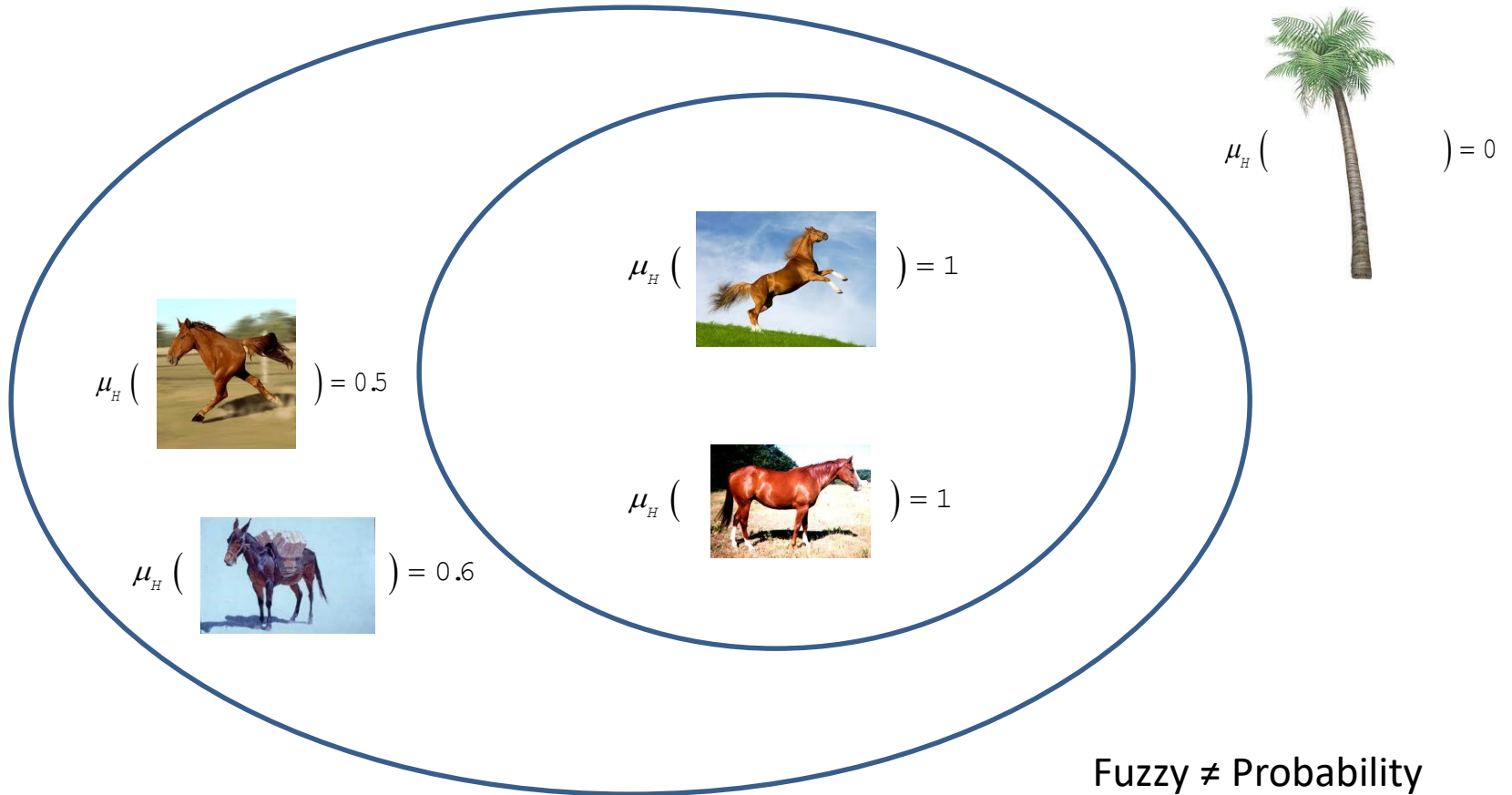
Is this a horse?



?
 $\in H$

Fuzzy sets

H – set of horses



Fuzzy concept of safety

$$\frac{P_{\max}}{P_{\text{design}}} = \gamma$$

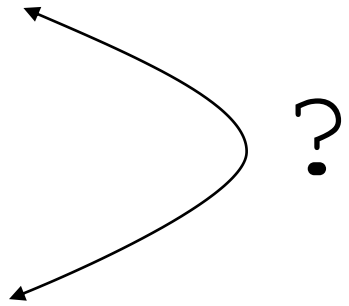
$$g(\mathbf{x}) = \gamma$$

$$P_f = P \{ g(\mathbf{x}) < 0 \} < P_f^0$$

Problems with binary logic

- Is it possible to find in the real world statements which are absolutely true?

(L. Wittgenstein, Tractatus Logico-Philosophicus, Annalen der Naturphilosophie, 14, 1921)



Modus ponens can be applied if $P \Rightarrow Q$ and Q are true.

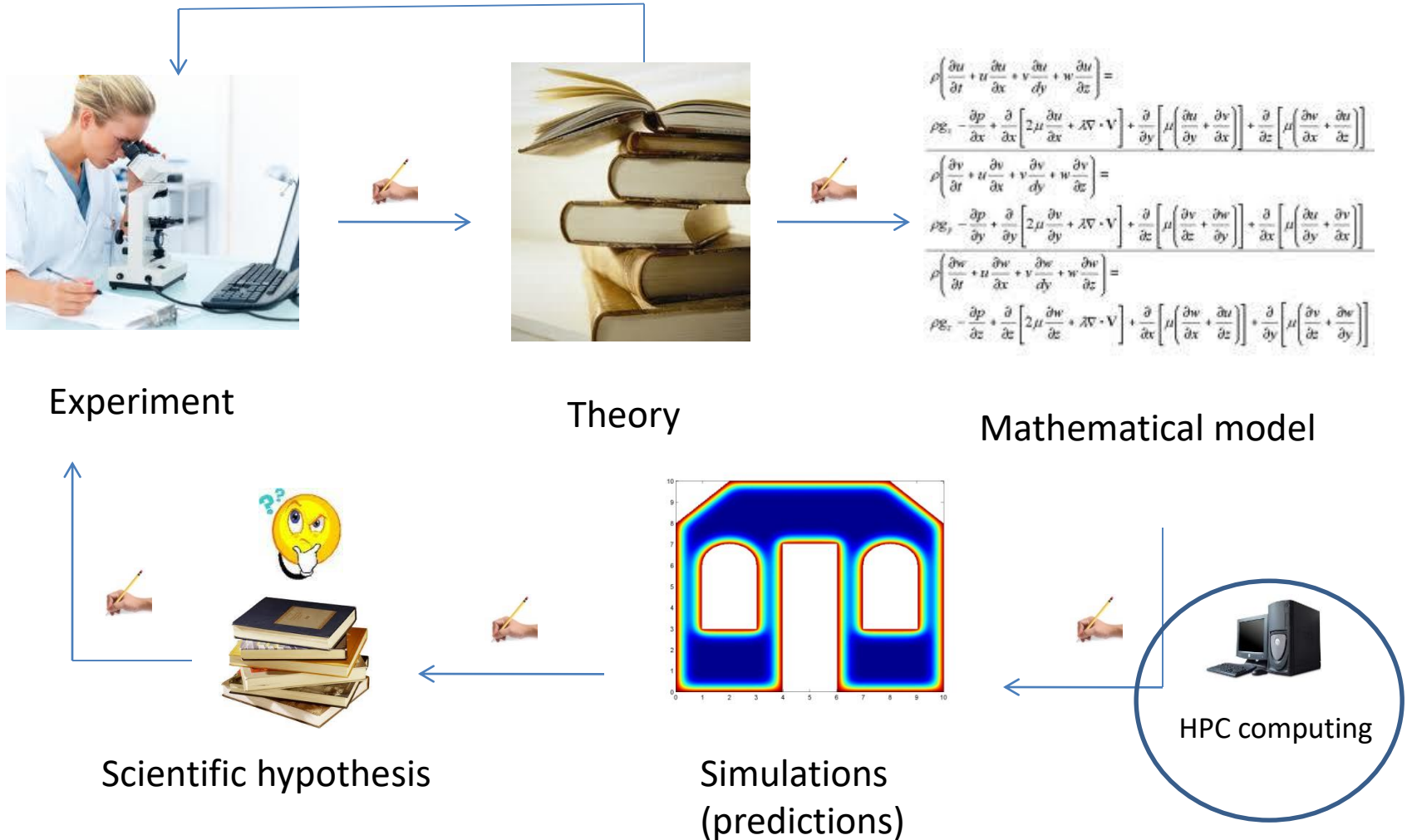


uncertainty.wmv

$$\frac{P \Rightarrow Q, P}{Q}$$

When modus ponens
can be applied?

Science



Mathematics and programming

Mathematics

mathematical method



results

Programming

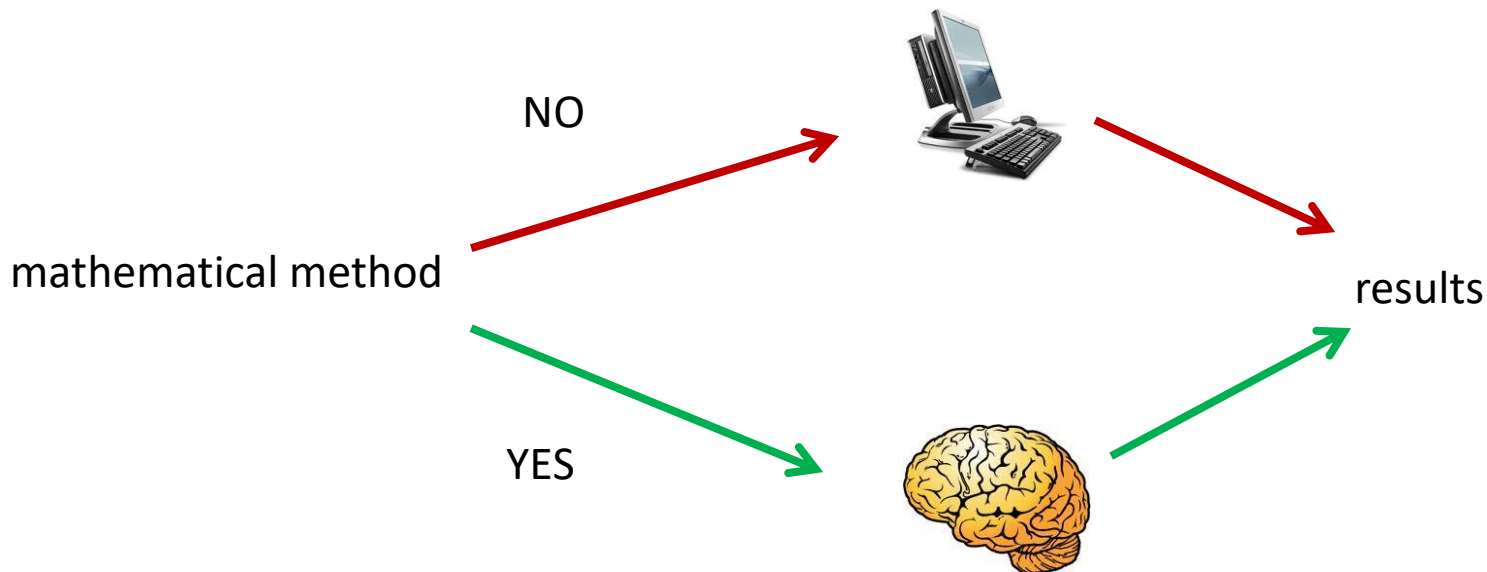
program



results

Main problem

- At this moment it is not possible perform general mathematical research automatically without human input.



Tools

- Approach without tools
- Approach with tools

5 years of training



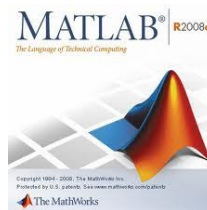
Final result



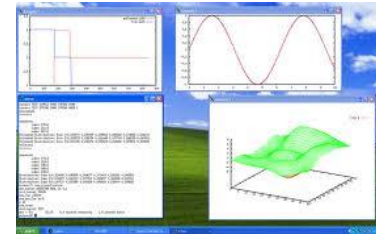
Mathematical tools



Mathematica



Matlab

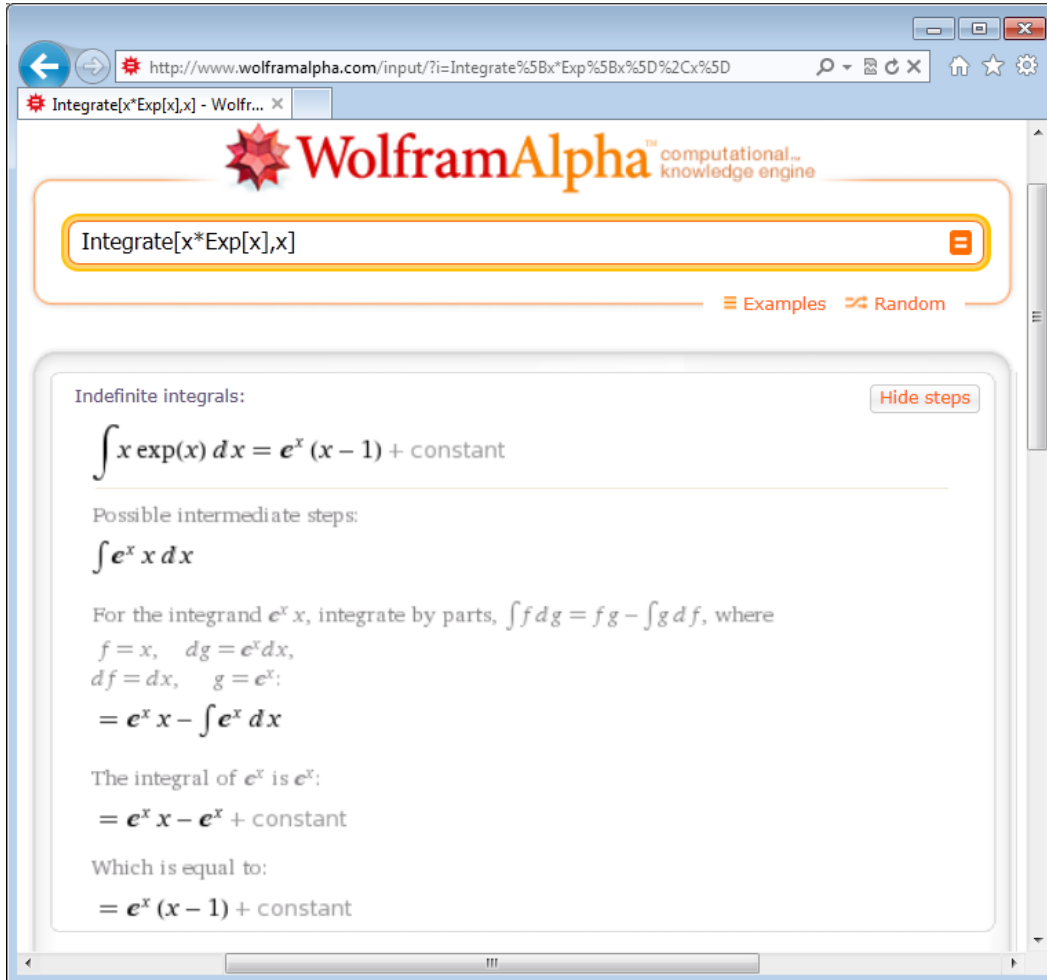


Octave

Etc.

Example:

<http://www.wolframalpha.com>



The screenshot shows the WolframAlpha web interface. The browser address bar displays the URL: http://www.wolframalpha.com/input/?i=Integrate%5Bx*Exp%5Bx%5D%2Cx%5D. The search bar contains the input `Integrate[x*Exp[x],x]`. Below the search bar, the results are displayed under the heading "Indefinite integrals:". The main result is $\int x \exp(x) dx = e^x (x - 1) + \text{constant}$. A "Hide steps" button is located to the right of this result. Below the main result, the section "Possible intermediate steps:" shows the integration process. It starts with $\int e^x x dx$, followed by the instruction "For the integrand $e^x x$, integrate by parts, $\int f dg = fg - \int g df$, where $f = x$, $dg = e^x dx$, $df = dx$, $g = e^x$:". The next line shows the result of integration by parts: $= e^x x - \int e^x dx$. Then, it states "The integral of e^x is e^x :" followed by $= e^x x - e^x + \text{constant}$. Finally, it concludes with "Which is equal to:" and the final result $= e^x (x - 1) + \text{constant}$.

It is possible to calculate not only the result but also intermediate steps in the calculations

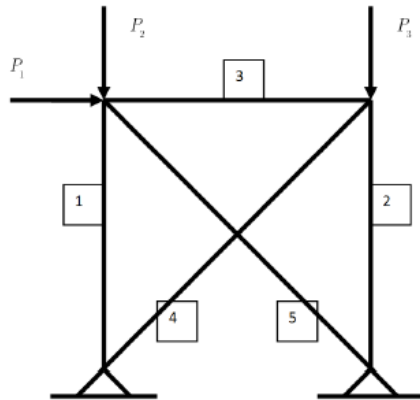
Example: student's tests

1 000 000 pages

current-test	1,638	09/12/2011	00:33-a-
math-2326-t1-q1-1	47,391	09/12/2011	14:25-a-
math-2326-t1-q1-2	47,186	09/12/2011	13:18-a-
math-2326-t1-q1-3	35,023	09/12/2011	00:01-a-
math-2326-t1-q1-4	35,170	09/12/2011	00:02-a-
math-2326-t1-q2-1	34,207	09/12/2011	00:09-a-
math-2326-t1-q2-2	33,940	09/12/2011	00:09-a-
math-2326-t1-q2-3	34,467	09/12/2011	00:10-a-
math-2326-t1-q2-4	34,968	09/12/2011	00:10-a-
math-2326-t1-q3-1	34,603	09/12/2011	00:14-a-
math-2326-t1-q3-2	34,403	09/12/2011	00:14-a-
math-2326-t1-q3-3	34,456	09/12/2011	00:14-a-
math-2326-t1-q3-4	34,508	09/12/2011	00:15-a-
math-2326-t1-q4-1	34,374	09/12/2011	00:16-a-
math-2326-t1-q4-2	34,669	09/12/2011	00:16-a-
math-2326-t1-q4-3	34,782	09/12/2011	00:17-a-
math-2326-t1-q4-4	34,834	09/12/2011	00:17-a-
math-2326-t1-q5-1	33,782	09/12/2011	00:20-a-
math-2326-t1-q5-2	33,834	09/12/2011	00:21-a-
math-2326-t1-q5-3	33,981	09/12/2011	00:22-a-
math-2326-t1-q5-4	34,195	09/12/2011	00:23-a-
template	37,548	07/21/2011	19:24-a-

[.]	<DIR>	07/30/2011	10:45—
test-0-5000	docx	3,182,160	07/29/2011 23:35-a-
test-100000-105000	docx	3,183,466	07/30/2011 00:23-a-
test-10000-15000	docx	3,183,239	07/29/2011 23:40-a-
test-105000-110000	docx	3,175,851	07/30/2011 00:26-a-
test-110000-115000	docx	3,181,359	07/30/2011 00:28-a-
test-115000-120000	docx	3,184,515	07/30/2011 00:31-a-
test-120000-125000	docx	3,182,228	07/30/2011 00:33-a-
test-125000-130000	docx	3,179,647	07/30/2011 00:36-a-
test-130000-135000	docx	3,179,880	07/30/2011 00:38-a-
test-135000-140000	docx	3,177,462	07/30/2011 00:41-a-
test-140000-145000	docx	3,174,780	07/30/2011 00:43-a-
test-145000-150000	docx	3,179,052	07/30/2011 00:46-a-
test-150000-155000	docx	3,182,749	07/30/2011 00:48-a-
test-15000-20000	docx	3,179,026	07/29/2011 23:42-a-
test-155000-160000	docx	3,186,131	07/30/2011 00:51-a-
test-160000-165000	docx	3,179,716	07/30/2011 00:53-a-
test-165000-170000	docx	3,176,107	07/30/2011 00:56-a-
test-170000-175000	docx	3,178,196	07/30/2011 00:59-a-
test-175000-180000	docx	3,180,116	07/30/2011 01:01-a-
test-180000-185000	docx	3,176,812	07/30/2011 01:03-a-
test-185000-190000	docx	3,177,382	07/30/2011 01:06-a-
test-190000-195000	docx	3,179,524	07/30/2011 01:08-a-
test-195000-200000	docx	3,182,868	07/30/2011 01:11-a-
test-200000-205000	docx	3,172,996	07/30/2011 01:13-a-
test-20000-25000	docx	3,181,452	07/29/2011 23:44-a-
test-205000-210000	docx	3,184,739	07/30/2011 01:16-a-
test-210000-215000	docx	3,177,692	07/30/2011 01:18-a-
test-215000-220000	docx	3,175,399	07/30/2011 01:21-a-
test-220000-225000	docx	3,181,219	07/30/2011 01:23-a-
test-225000-230000	docx	3,185,913	07/30/2011 01:26-a-
test-230000-235000	docx	3,184,646	07/30/2011 01:28-a-
test-235000-240000	docx	3,176,807	07/30/2011 01:30-a-
test-240000-245000	docx	3,172,987	07/30/2011 01:33-a-
test-245000-250000	docx	3,181,728	07/30/2011 01:35-a-
test-250000-255000	docx	3,180,128	07/30/2011 01:37-a-
test-25000-30000	docx	3,182,087	07/29/2011 23:47-a-
test-255000-260000	docx	3,183,192	07/30/2011 01:40-a-
test-260000-265000	docx	3,177,516	07/30/2011 01:42-a-
test-265000-270000	docx	3,182,184	07/30/2011 01:44-a-
test-270000-275000	docx	3,183,853	07/30/2011 01:47-a-
test-275000-280000	docx	3,177,966	07/30/2011 01:49-a-

Automated reports in Latex



200 pages

Table 260: Interval data (uncertainty 37%)

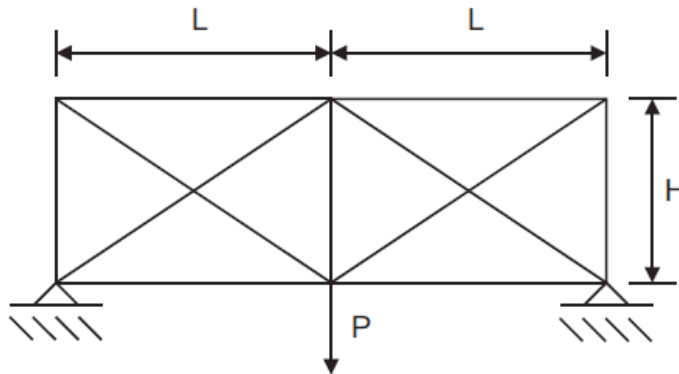
Name	Value	Units
E	[126000000000, 274000000000]	$\frac{N}{m^2}$
A	0.0001	m^2
P	[630, 1370]	N
L	1	m
H	1	m

Table 261: Interval displacements (uncertainty 37%).

u	$\underline{u}[m]$	$\text{mid}(u)[m]$	$\bar{u}[m]$
1	1.21305600352052E-05	0.000100909065484165	0.000252372472465261
2	-6.85096385258009E-05	-1.05605747310197E-05	4.59064530921089E-05
3	1.301976476875E-05	9.0433907531449E-05	0.000230202461510371
4	-0.000147816134050454	-6.05605747310197E-05	-2.12951211825158E-05

Table 262: Combinations (uncertainty 37%, -1 - lower bound, 1 - upper bound, 0 - mid point solution).

u	E_1	E_2	E_3	E_4	E_5	P_1	P_2	P_3
1 (inf)	-1	1	-1	-1	1	-1	1	-1
1 (sup)	-1	-1	-1	-1	1	1	-1	1
monotone	no	yes	no	no	yes	yes	yes	yes
2 (inf)	-1	1	1	1	-1	-1	1	-1
2 (sup)	-1	-1	-1	-1	1	1	-1	1
monotone	no	yes	yes	yes	yes	yes	yes	yes
3 (inf)	-1	1	1	-1	-1	1	1	-1
3 (sup)	-1	-1	1	-1	-1	1	-1	1
monotone	no	yes	no	no	yes	yes	yes	yes
4 (inf)	-1	-1	1	1	-1	1	-1	1
4 (sup)	-1	1	1	1	1	-1	1	-1
monotone	no	yes	no	no	yes	yes	yes	yes



550 pages

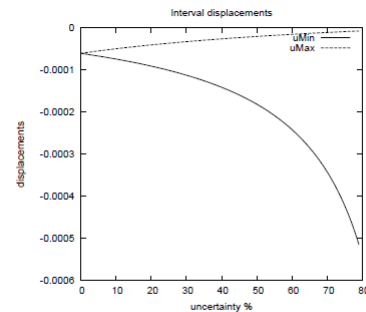


Figure 4: Uncertainty in displacements: u_2



Plate equation

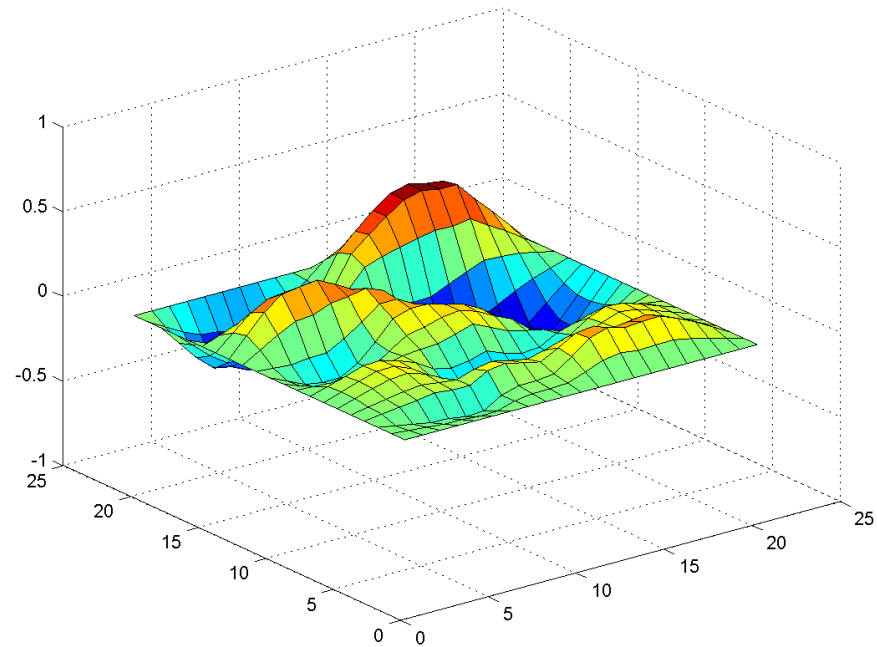
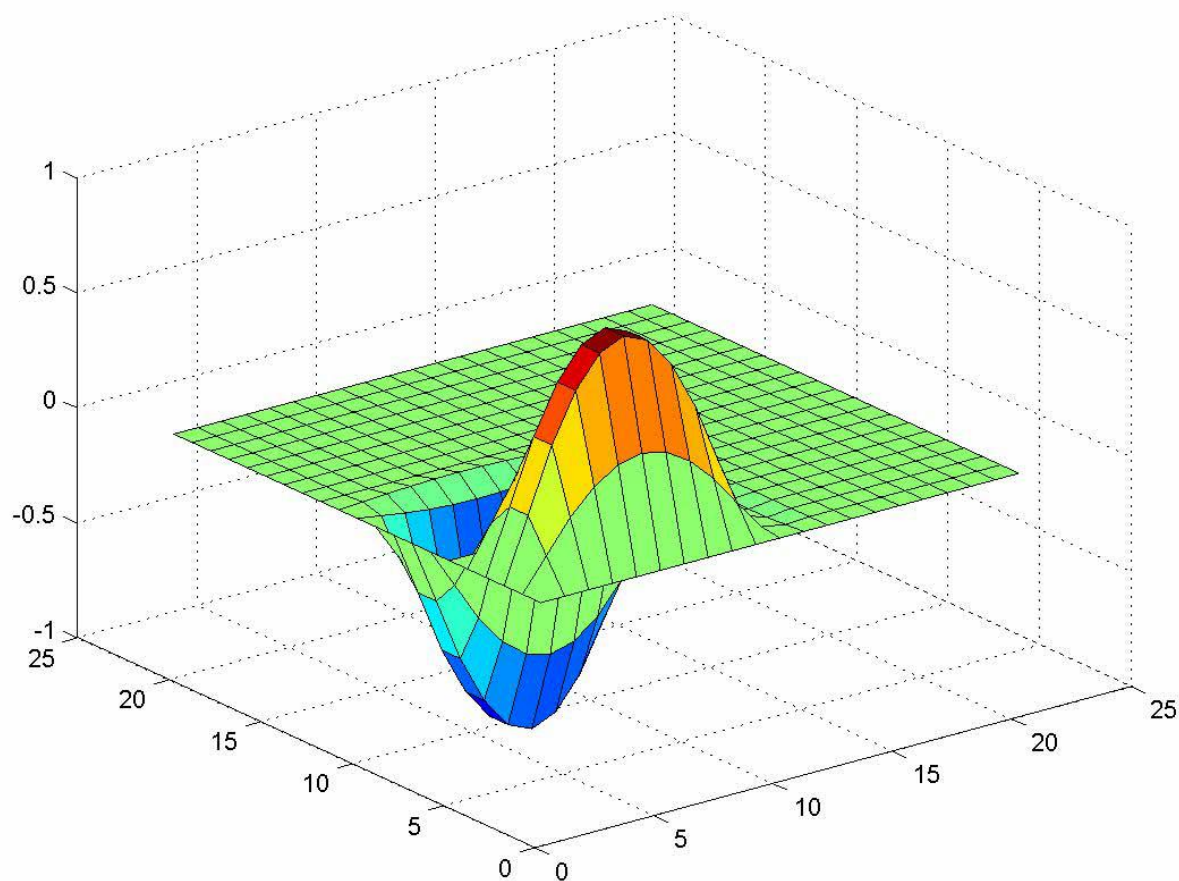


Plate equation



Thank you

I will be back ...
with new results soon

