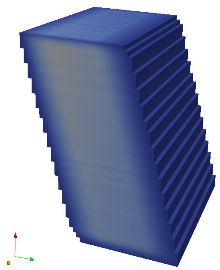
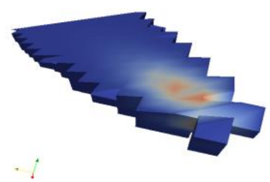
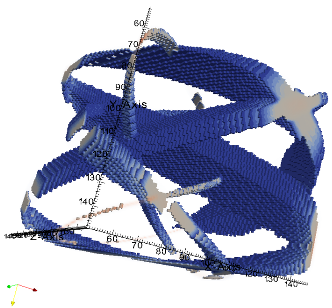
Linear Dynamics of an Elastic Beam and Plate Under Moving Loads with Uncertain Parameters







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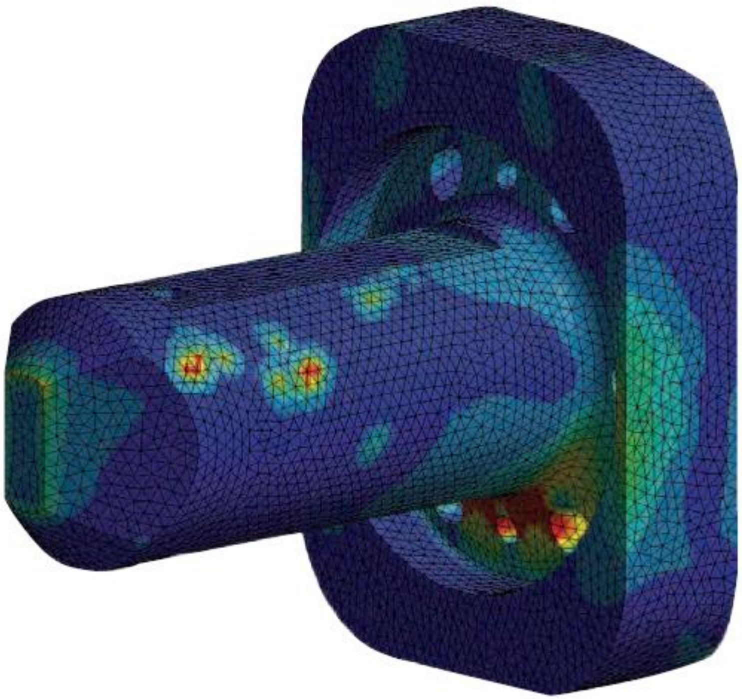
http://andrzej.pownuk.com

Outline of the presentation

- Equations with the uncertain parameters and their applications
- New approach for the solution of the equations with the interval parameters
- Generalizations and conclusions

Mathematical model of a machine



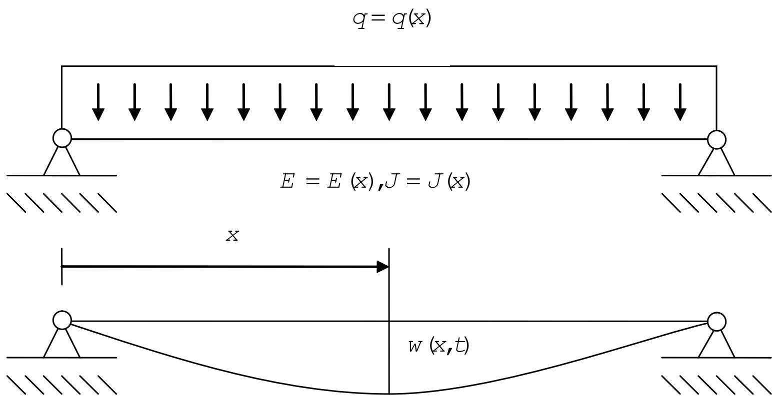
$$
\begin{cases}
\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \\
\sigma_{ij} = \sum_{k=1}^{3} \sum_{j=1}^{3} C^{ijkl} \varepsilon_{kl} \\
\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
u_i = u_i^*, x \in \partial V_u \\
\sum_{j=1}^{3} \sigma_{ij} n_j = t_i^*, x \in \partial V_\sigma \\
u_i = u_i(x,0), x \in V
\end{cases}
$$

Such simulations are possible since early 1970s

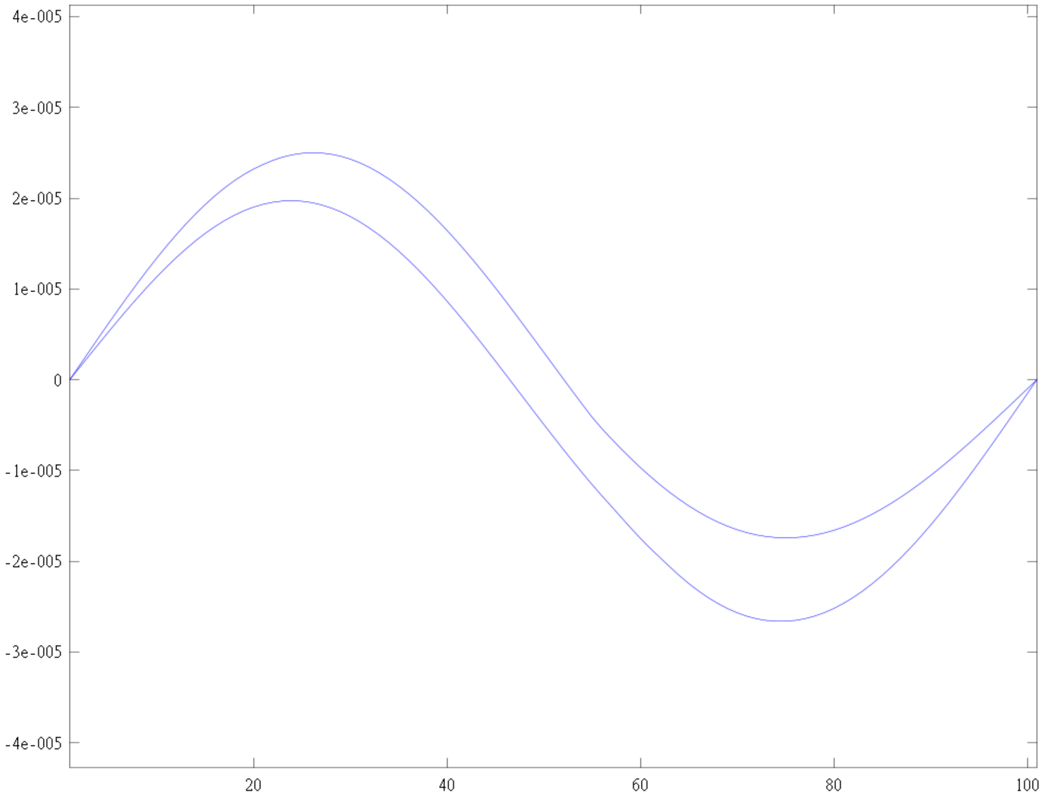
O.C. Zienkiewicz, Ivo M. Babuška, P.G. Ciarlet ...

- Beam model with the interval parameters

$$
\begin{cases}
EJ\dfrac{\partial^4 w}{\partial x^4} = q - \rho A\dfrac{\partial^2 w}{\partial t^2} \\
w(0,t) = 0 \\
w(L,t) = 0 \\
\dfrac{\partial^2 w(0,t)}{dx^2} = 0 \\
\dfrac{\partial^2 w(L,t)}{dx^2} = 0 \\
w(x,0) = w_0(x) \\
v(x,0) = \dfrac{\partial w(x,0)}{\partial t} = v_0(x) \\
E \in \mathbf{E}, q \in \mathbf{q}, A \in \mathbf{A}
\end{cases}
$$

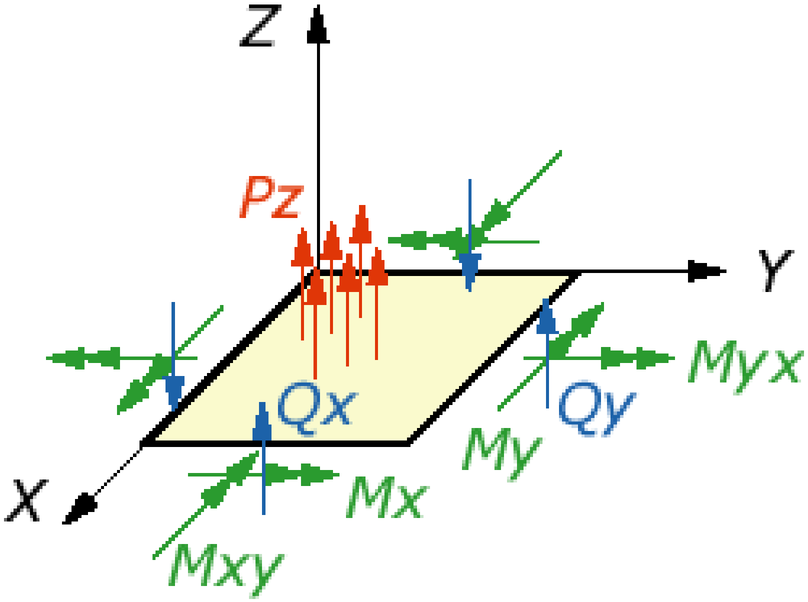


Interval displacements



- Plate with the interval parameters

$$\begin{cases} D\left(\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}\right)=q-\rho h\frac{\partial^2 u}{\partial t^2} \\ u(0,y,t)=0 \\ u(L,y,t)=0 \\ u(x,0,t)=0 \\ u(x,L,t)=0 \\ \frac{\partial^2 u}{\partial x^2}(0,y,t)=0 \\ \frac{\partial^2 u}{\partial x^2}(L,y,t)=0 \\ \frac{\partial^2 u}{\partial y^2}(x,0,t)=0 \\ \frac{\partial^2 u}{\partial y^2}(x,L,t)=0 \\ u(x,y,0)=u^*(x,y) \\ \frac{\partial u(x,y,0)}{\partial t}=v^*(x,y) \\ E\in\mathbf{E},q\in\mathbf{q},h\in\mathbf{h} \end{cases}$$



Mathematical models

physical problem

$$\begin{aligned}\sigma_{ij,j} + F_i &= 0 \text{ in } \Omega,\\ \epsilon_{ij} &= \frac{1}{2}(u_{j,i} + u_{i,j}),\\ \sigma_{ij} &= C_{ijkl}\,\epsilon_{kl}.\end{aligned}$$

mathematical models

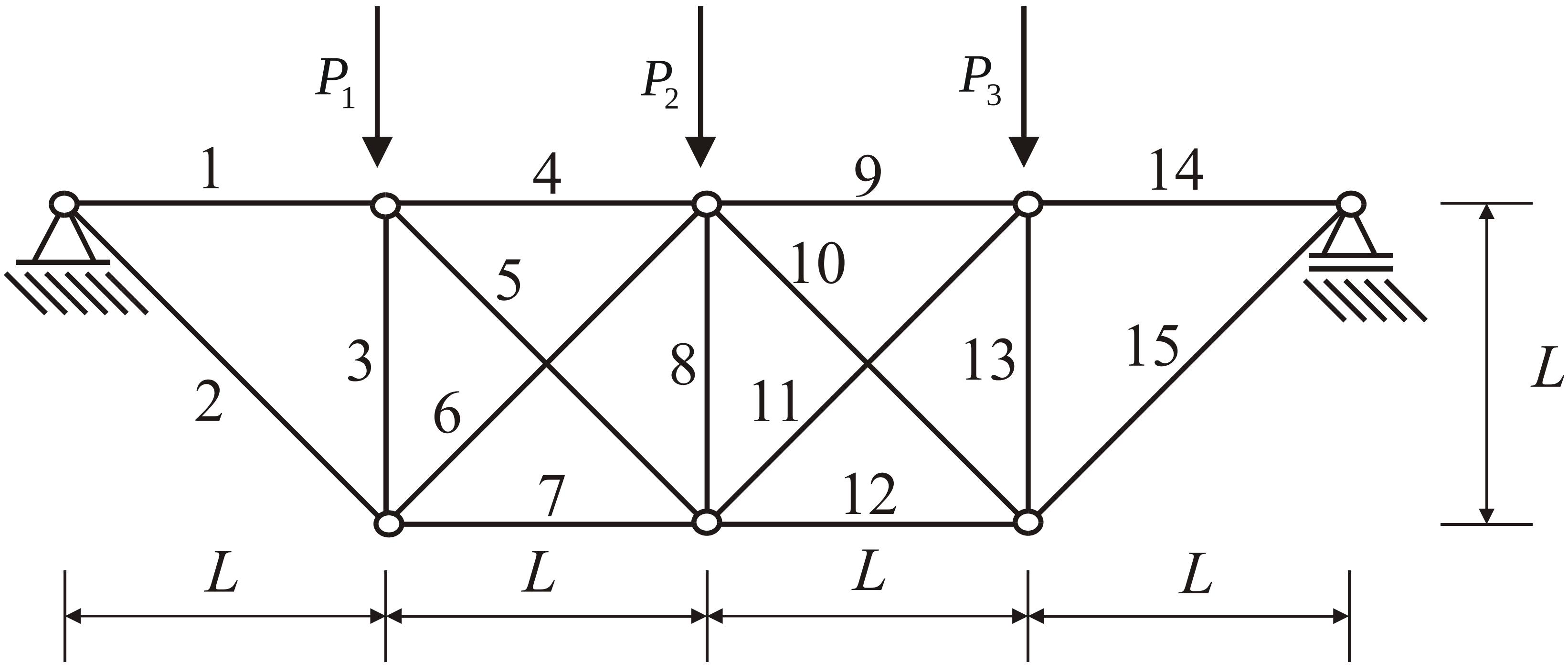
cheap

expensive

experiments

experimental results $\approx$ predictions

Truss structure with uncertain forces

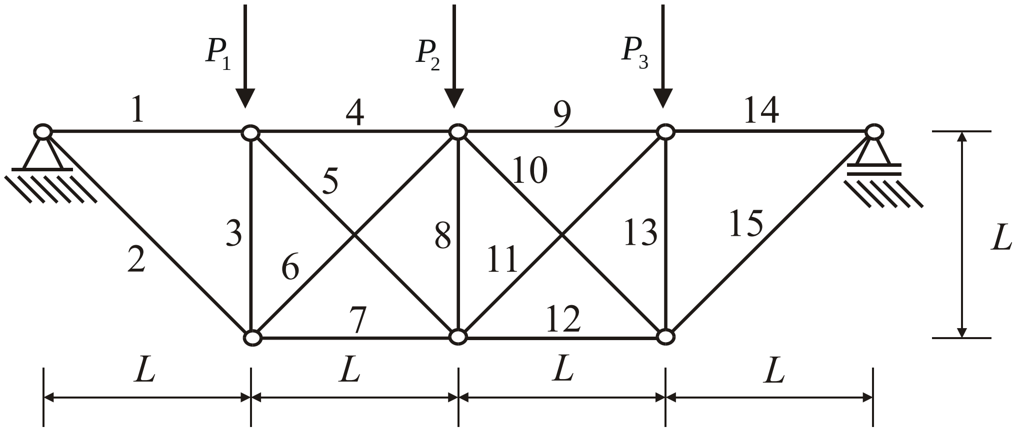


Perturbated forces

$$P = P_0 \pm \Delta P$$

5% uncertainty

No	1	2	3	4	5	6	7	8
ERROR %	10	9,998586	10,00184	10,00126	60,18381	11,67825	9,998955	31,8762
No	9	10	11	12	13	14	15	
ERROR %	10,00126	11,67825	60,18381	9,998955	10,00184	10	9,998586	



Uncertainty

Problem with real parameters

$$2x = 4$$

$$x = \frac{4}{2} = 2$$

$$2 \in [1,3]$$

$$4 \in [3,5]$$

Problem with interval parameters

$$[1,3]x = [3,5]$$

$$x = ?$$

Algebraic Solution

$$[1,2]x = [1,4]$$

$$x = [1,2]$$

because

$$[1,2] \cdot [1,2] = [1,4]$$

United Solution Set

$$[1,2]x = [1,4]$$

$$\mathbf{x} = \frac{[1,4]}{[1,2]} = \left[\frac{1}{2}, 4\right]$$

because

$$\mathbf{x} = \{x : ax = b, a \in [1,2], b \in [1,4]\}$$

Comparison of the solution sets

$$[1,2]x = [1,4]$$

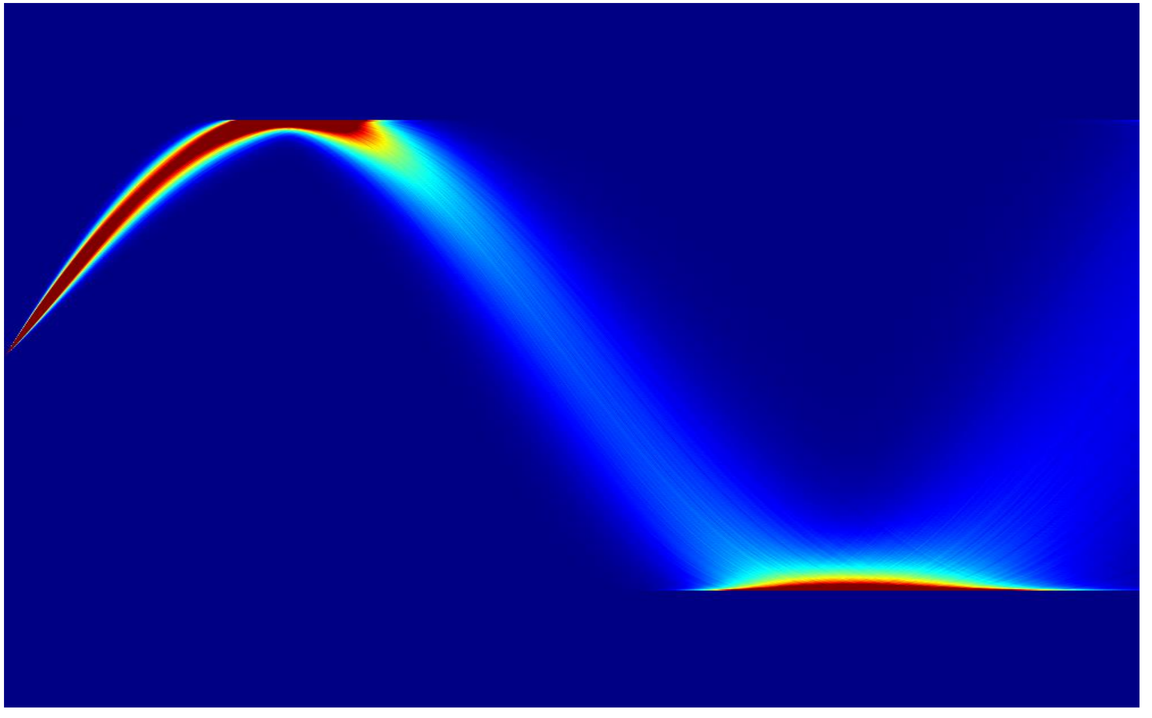
$$\mathbf{x} = [1,2] \qquad \neq \qquad \mathbf{x} = \frac{[1,4]}{[1,2]} = \left[\frac{1}{2}, 4\right]$$

Algebraic Solution **United Solution Set**

There are many ways how it is possible
to extend equations with the real parameters
into equations with the interval parameters.

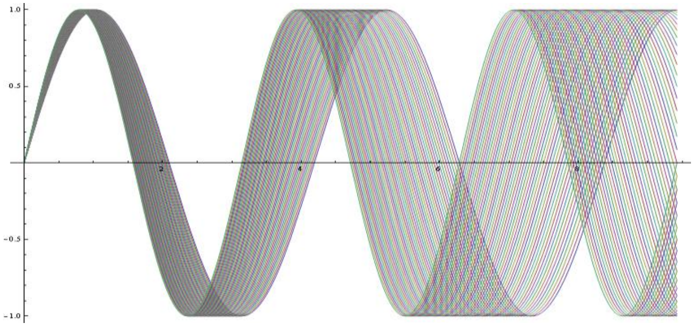
Stochastic differential equations

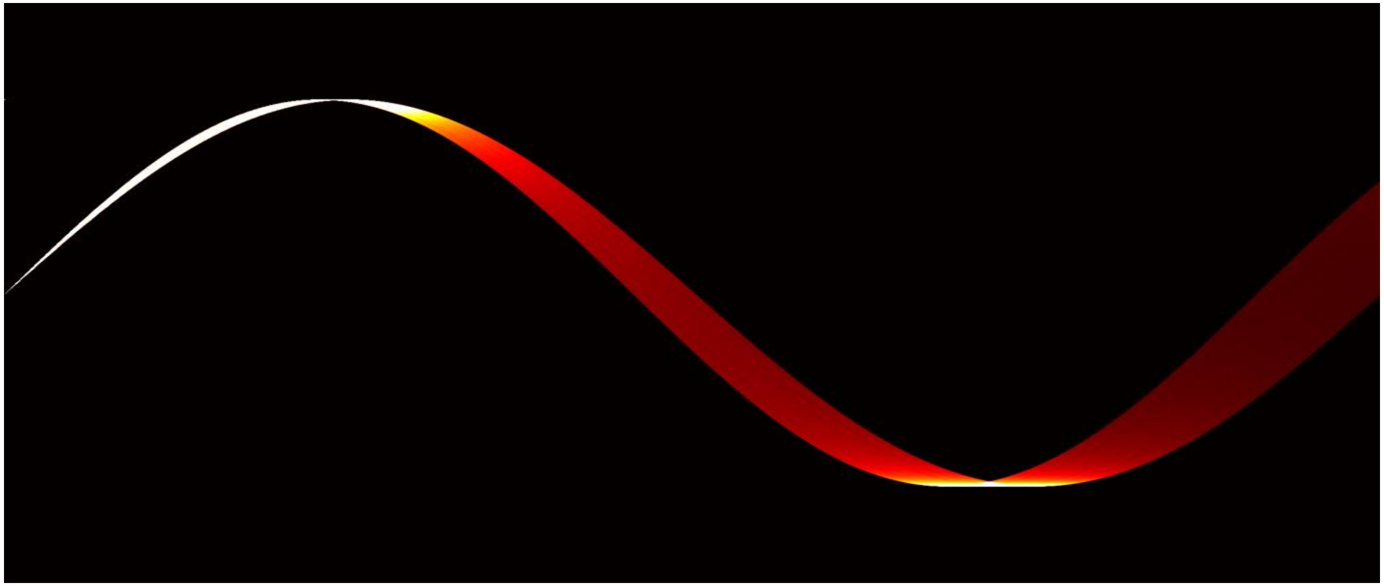
$$\begin{cases} y' = p\cos(pt) \\ y(0) = 0 \\ p \sim N(0,1) \end{cases}$$



Interval equation

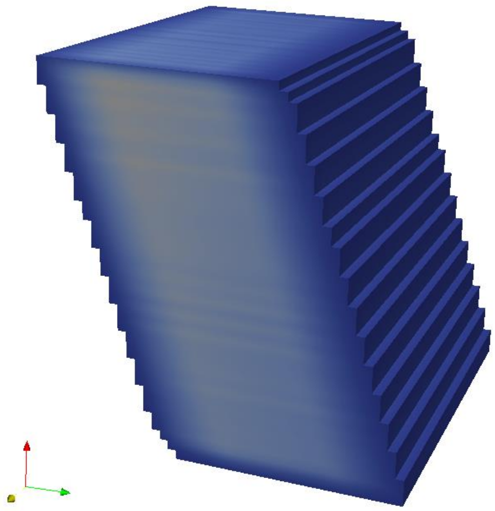
$$\begin{cases} y' = p\cos(pt) \\ y(0) = 0 \\ p \in \left[\underline{p}, \overline{p}\right] \end{cases}$$

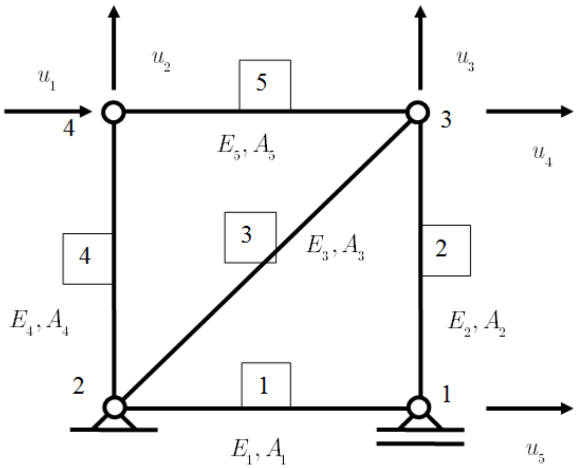




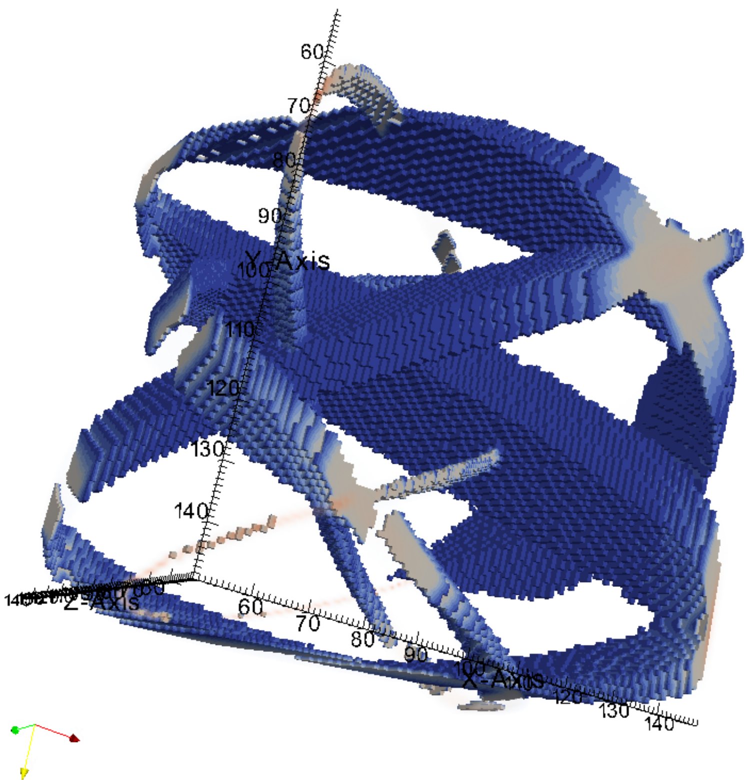
Solution set in 3D

$$u(\mathbf{p}) = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \dfrac{2\sqrt{2}P_1}{A_3E_3} + \dfrac{P_1}{A_5E_5} + \dfrac{P_1 + P_3}{A_2E_2} \\ -\dfrac{P_2}{A_4E_4} \\ -\dfrac{P_1 + P_3}{A_2E_2} \end{bmatrix} : E_i \in \left[\underline{E}_i, \overline{E}_i\right], P_i \in \left[\underline{P}_i, \overline{P}_i\right] \right\}$$



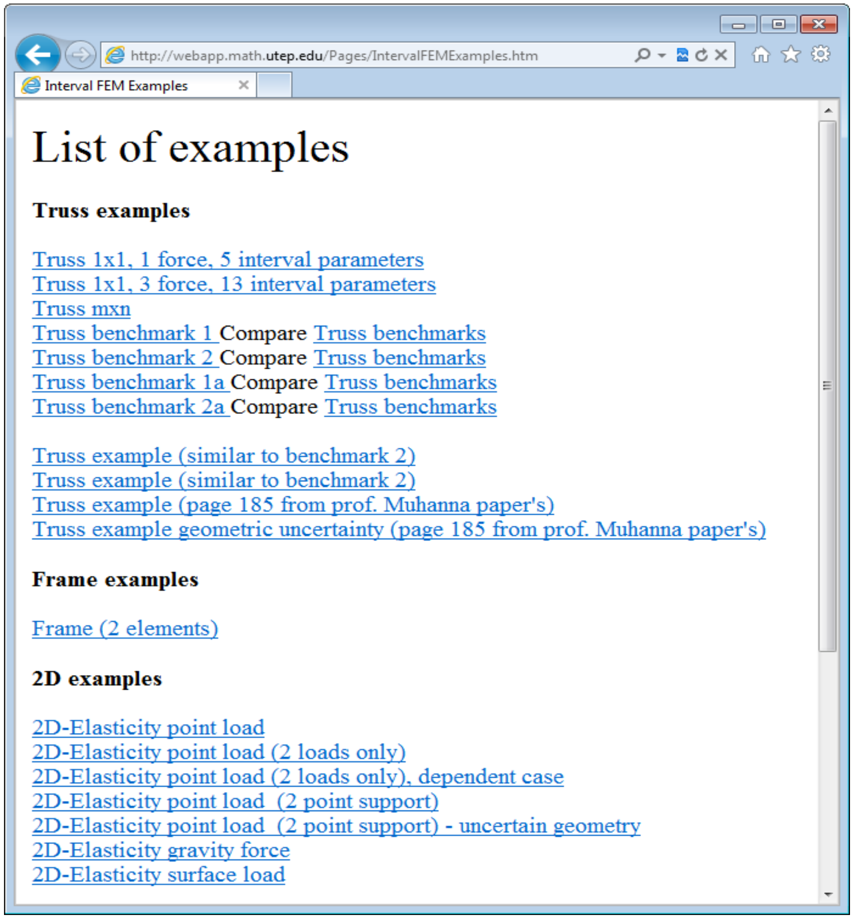


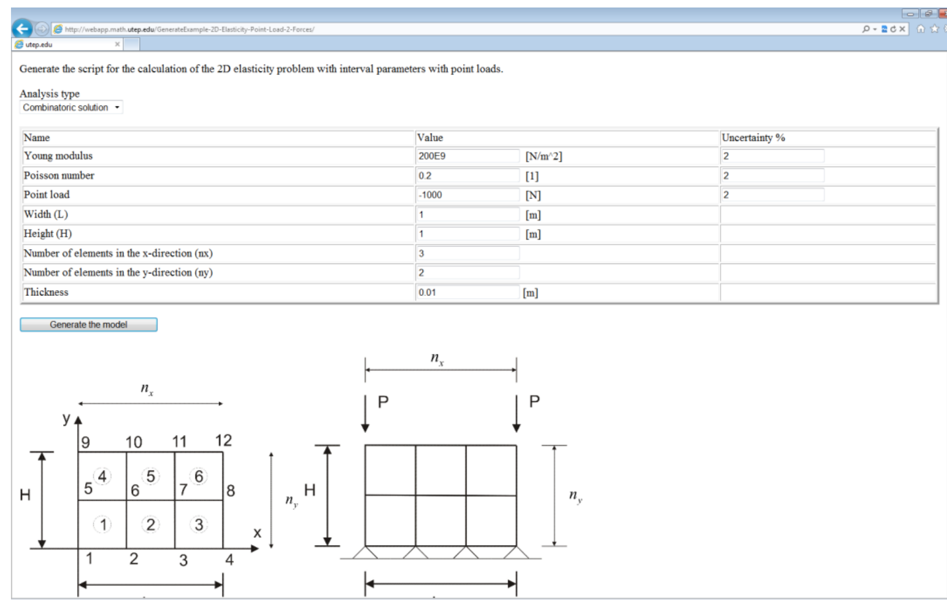
Solution set in 3D



$$\begin{cases} \dfrac{dx}{dt} = pr\cos(pt) \\ \dfrac{dy}{dt} = -pr\sin(pt) \\ \dfrac{dx}{dt} = p \\ x(0) = 0 \\ y(0) = 1 \\ z(0) = 0 \end{cases}, p \in [\underline{p}, \overline{p}], r \in [\underline{r}, \overline{r}]$$

Automatically generated test problems

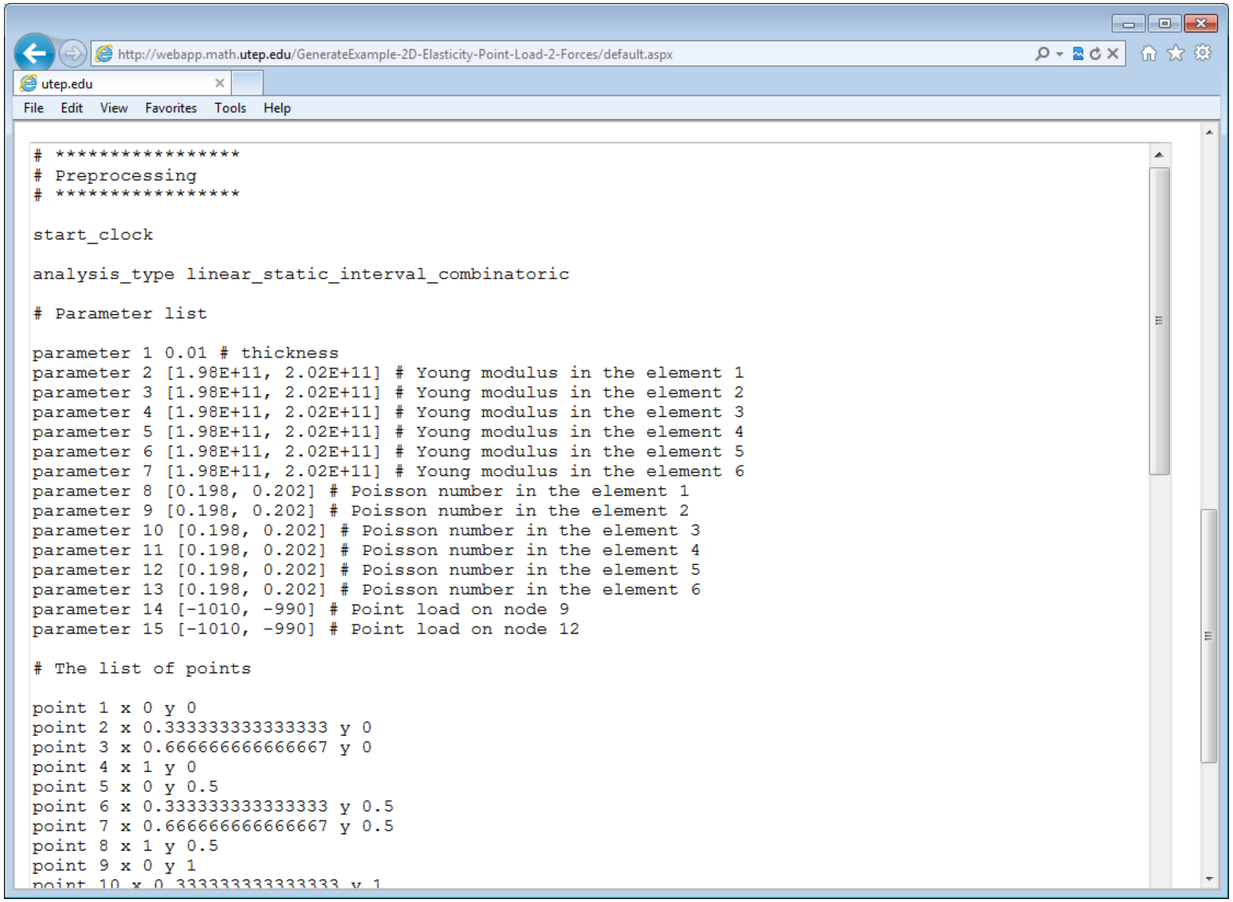




http://webapp.math.utep.edu/Pages/IntervalFEMExamples.htm

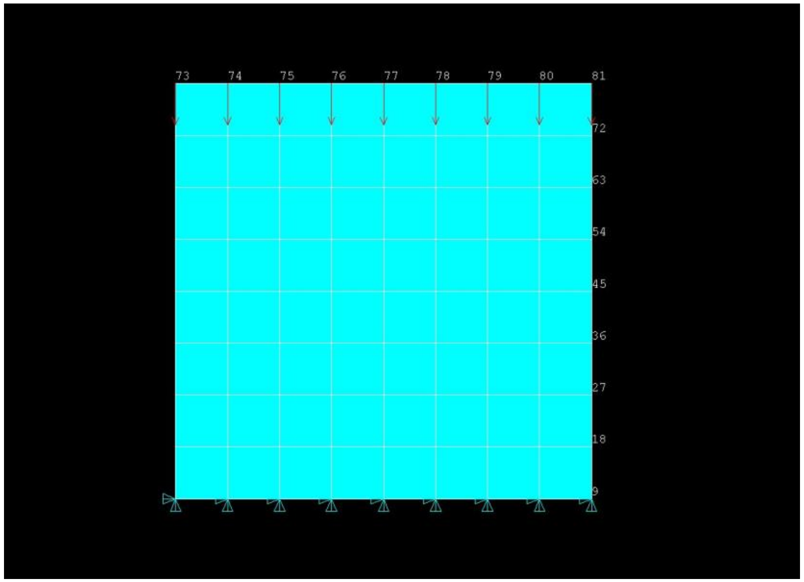
Automatically generated test problems

DSL
(Domain Specific Languages)

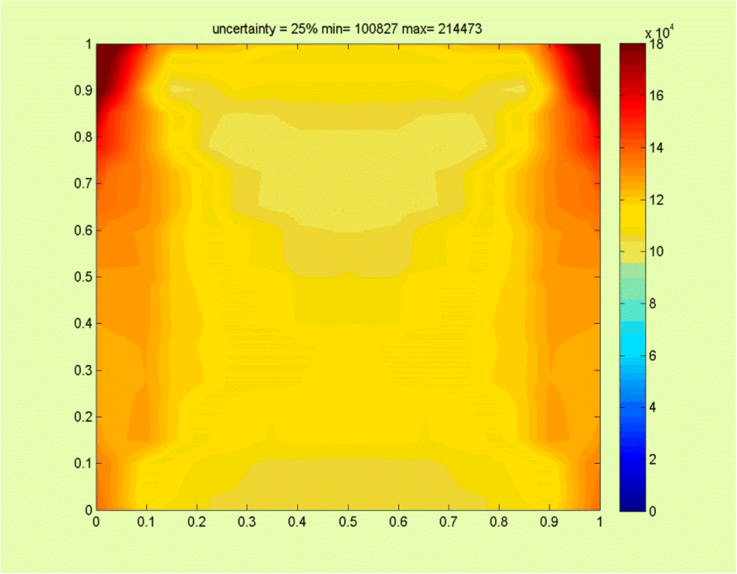


2D elasticity problem with the interval parameters

Model



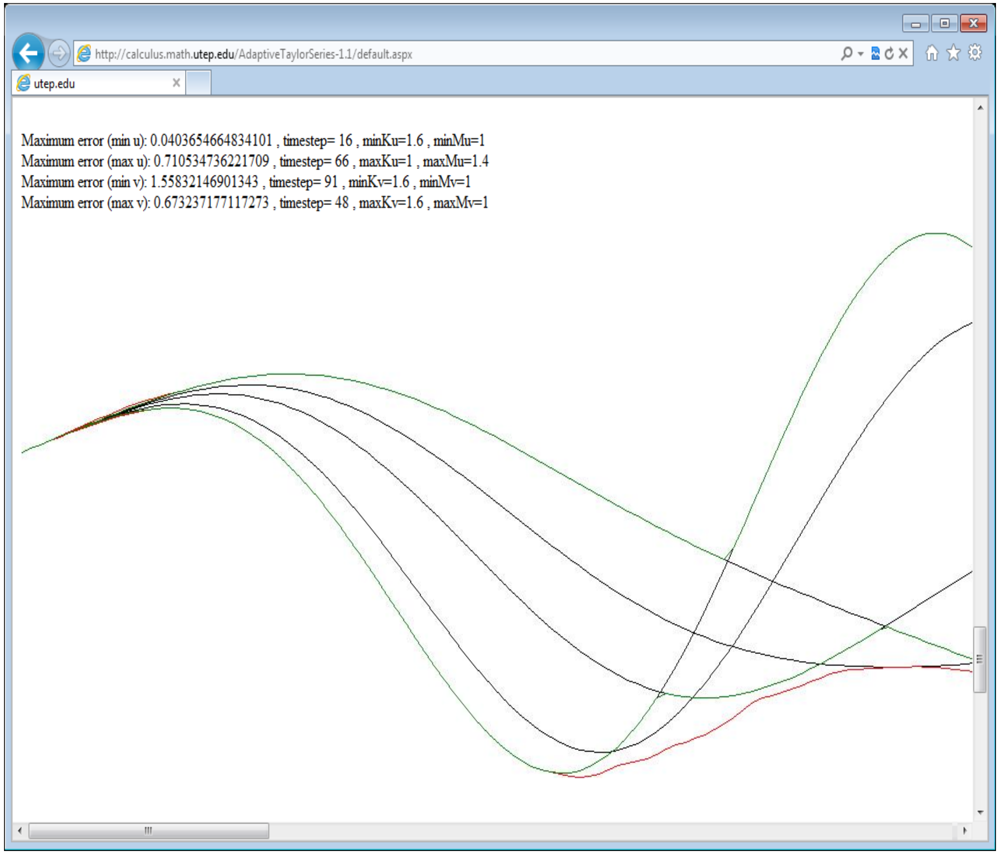
Solution



Mathematical model

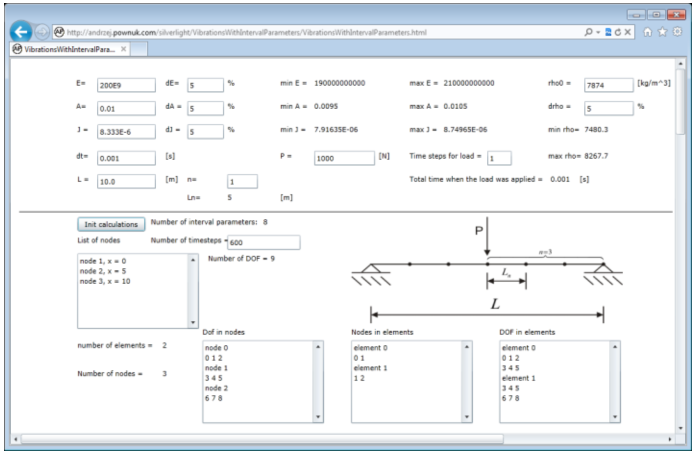
$$\mu u_{i,jj} + (\mu + \lambda) u_{j,ij} + F_i = \rho \partial_{tt} u_i \quad \text{or} \quad \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}) + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}.$$

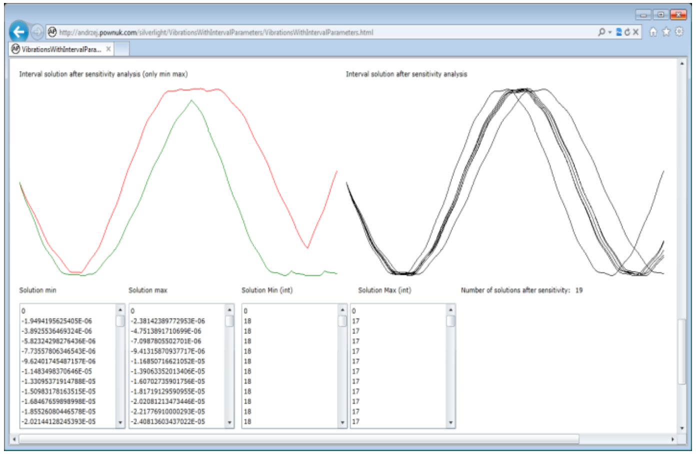
Adaptive Taylor series



http://webapp.math.utep.edu/AdaptiveTaylorSeries-1.1/

Adaptive Taylor series





http://andrzej.pownuk.com/silverlight/VibrationsWithIntervalParameters/VibrationsWithIntervalParameters.html

Tools which support my research

Epistemic uncertainty

H – set of horses

This is a horse.

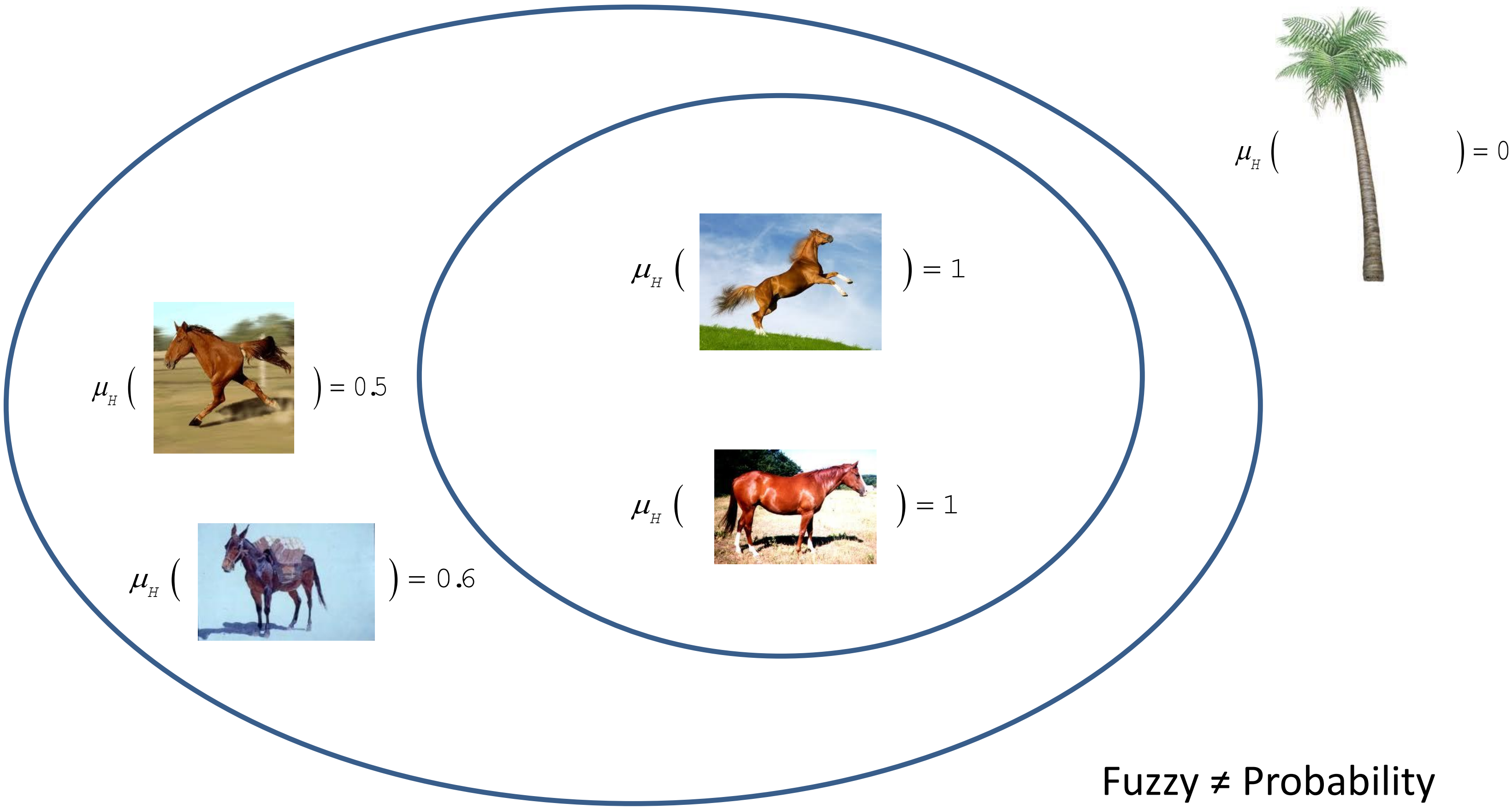
$\in H$

Is this a horse?

$\overset{?}{\in} H$

Fuzzy sets

H – set of horses



Fuzzy ≠ Probability

Fuzzy concept of safety

$$\frac{P_{\max}}{P_{design}} = \gamma$$

$$g(x) = \gamma$$

$$P_f = P\left\{g(x) < 0\right\} < P_f^0$$

Problems with binary logic

- Is it possible to find in the real world statements which are absolutely true?

(L. Wittgenstein, Tractatus Logico-Philosophicus, Annalen der Naturphilosophie, 14, 1921)

?

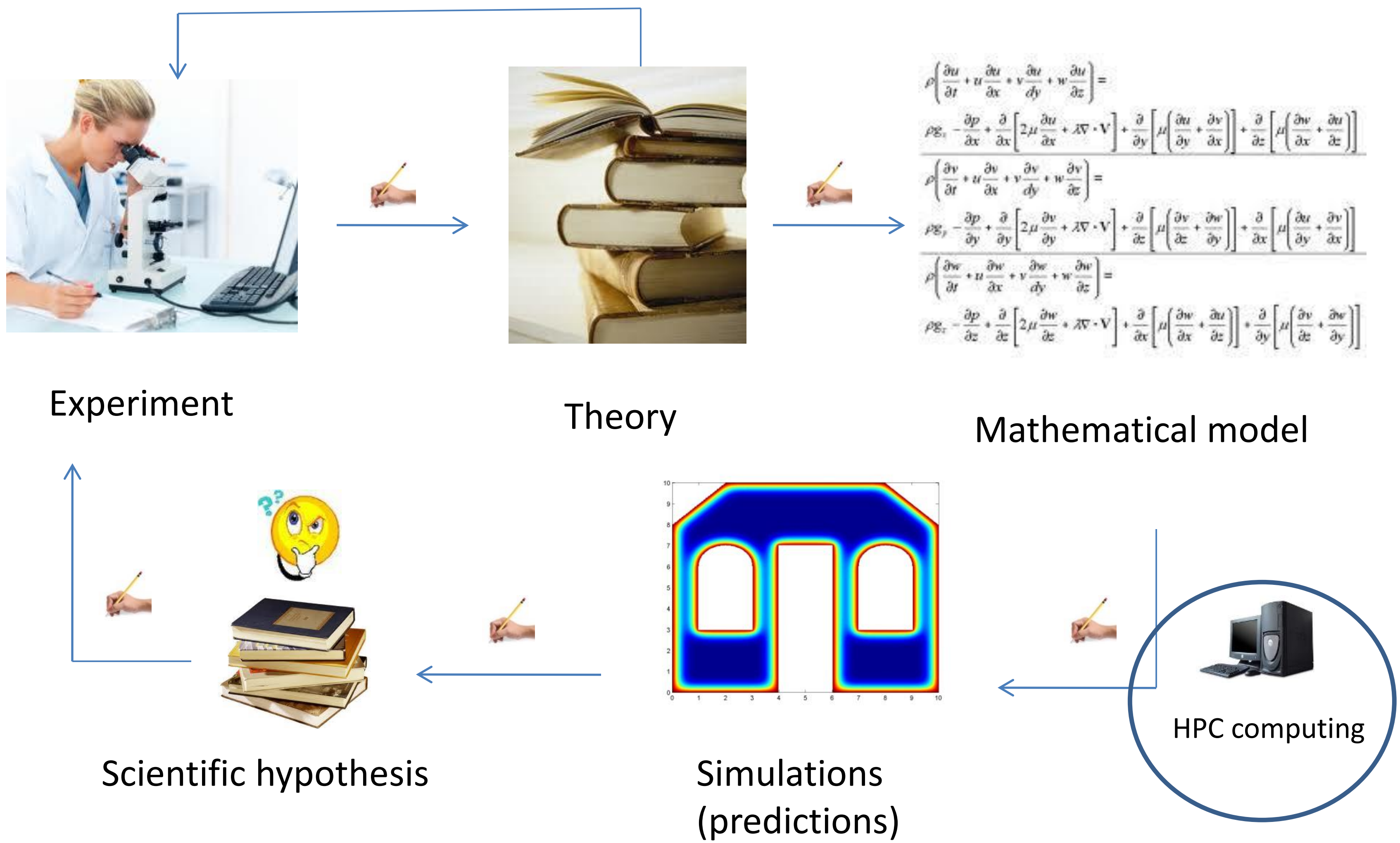
Modus ponens can be applied if $P \Rightarrow Q$ and Q are true.

uncertainty.wmv

$$\frac{P \Rightarrow Q , P}{Q}$$

When modus ponens can be applied?

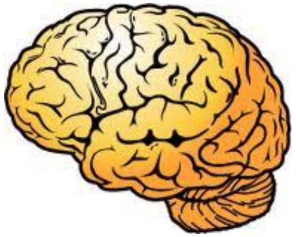
Science


Experiment
Theory
Mathematical model
Scientific hypothesis
Simulations
(predictions)
HPC computing

Mathematics and programming

Mathematics

mathematical method



results

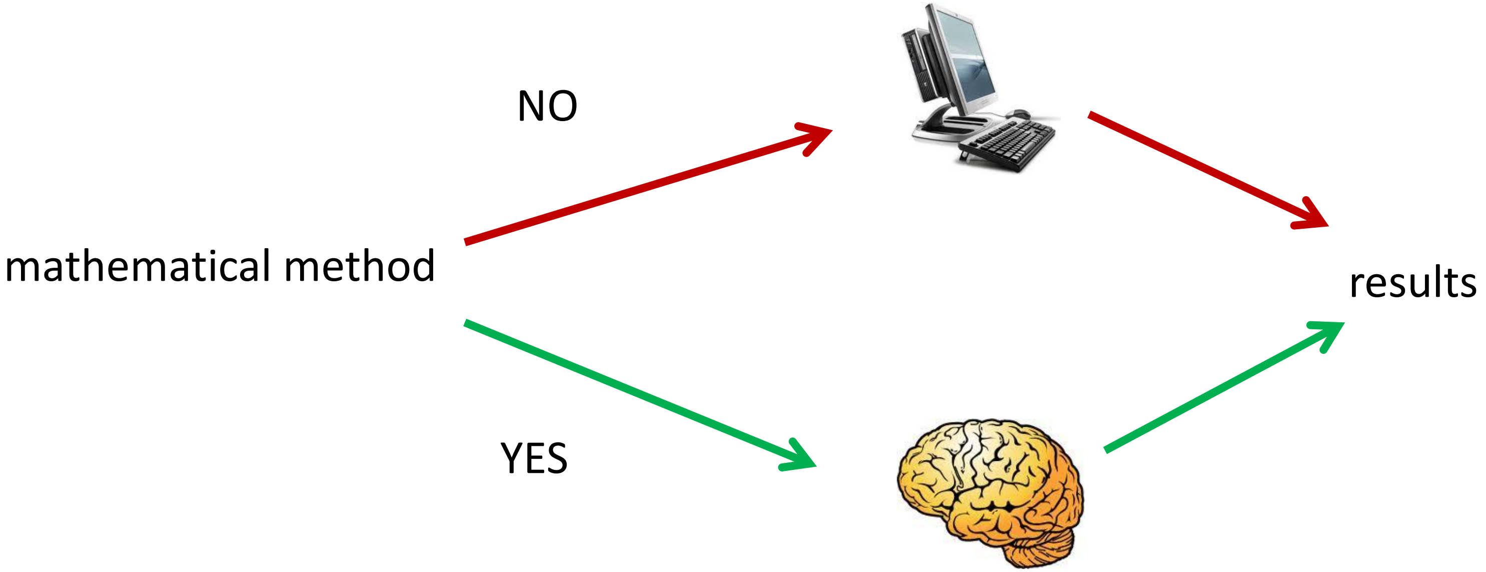
Programming

program

results

Main problem

- At this moment it is not possible perform general mathematical research automatically without human input.



Tools

- Approach without tools

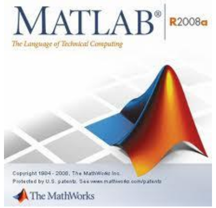
5 years of training

Final result

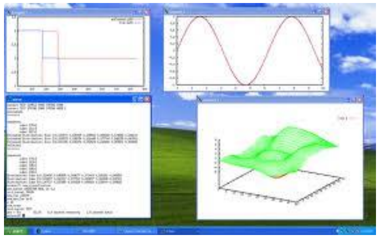
- Approach with tools

Mathematical tools

Mathematica



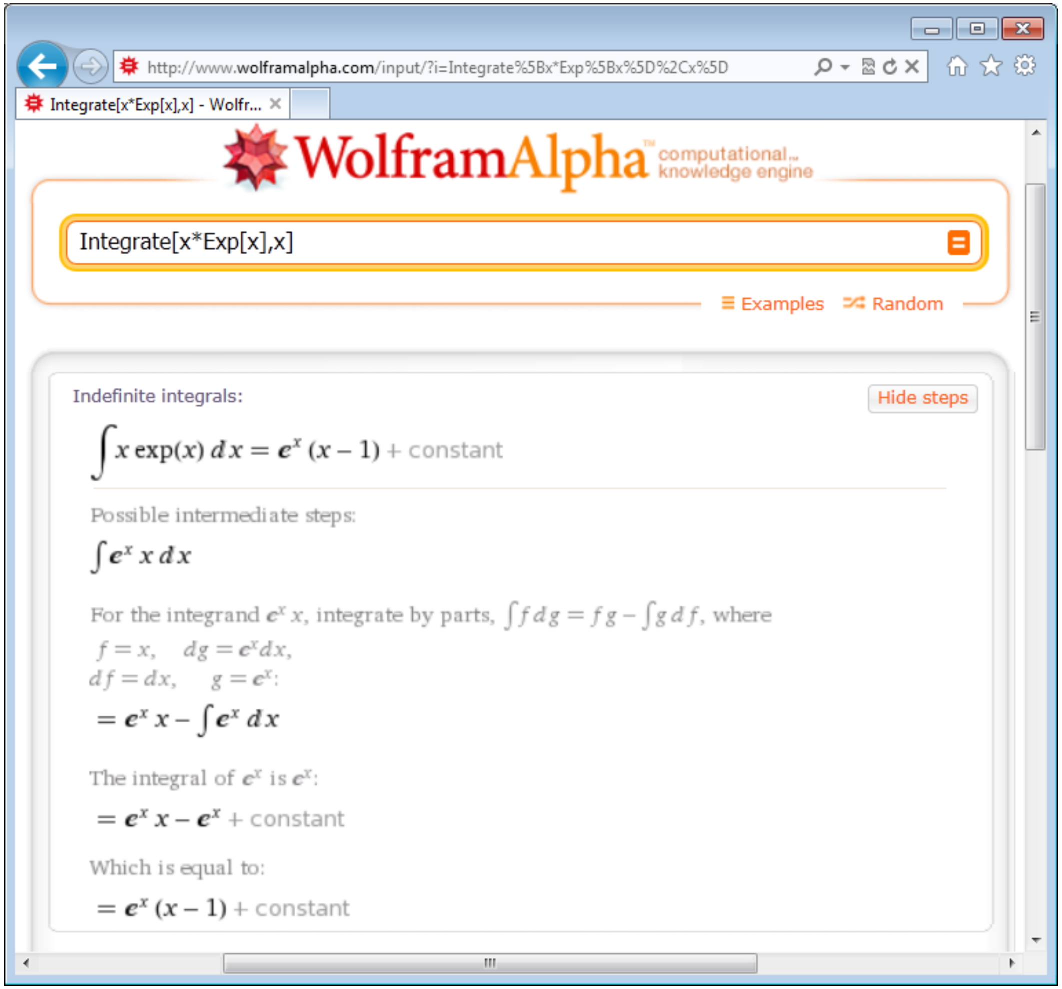
Matlab



Octave

Etc.

Example: http://www.wolframalpha.com



It is possible to calculate not only the result but also intermediate steps in the calculations

Example: student's tests

1 000 000 pages

current-test	1,638	09/12/2011	00:33	a–
math-2326-t1-q1-1	47,391	09/12/2011	14:25	a–
math-2326-t1-q1-2	47,186	09/12/2011	13:18	a–
math-2326-t1-q1-3	35,023	09/12/2011	00:01	a–
math-2326-t1-q1-4	35,170	09/12/2011	00:02	a–
math-2326-t1-q2-1	34,207	09/12/2011	00:09	a–
math-2326-t1-q2-2	33,940	09/12/2011	00:09	a–
math-2326-t1-q2-3	34,467	09/12/2011	00:10	a–
math-2326-t1-q2-4	34,968	09/12/2011	00:10	a–
math-2326-t1-q3-1	34,603	09/12/2011	00:14	a–
math-2326-t1-q3-2	34,403	09/12/2011	00:14	a–
math-2326-t1-q3-3	34,456	09/12/2011	00:14	a–
math-2326-t1-q3-4	34,508	09/12/2011	00:15	a–
math-2326-t1-q4-1	34,374	09/12/2011	00:16	a–
math-2326-t1-q4-2	34,669	09/12/2011	00:16	a–
math-2326-t1-q4-3	34,782	09/12/2011	00:17	a–
math-2326-t1-q4-4	34,834	09/12/2011	00:17	a–
math-2326-t1-q5-1	33,782	09/12/2011	00:20	a–
math-2326-t1-q5-2	33,834	09/12/2011	00:21	a–
math-2326-t1-q5-3	33,981	09/12/2011	00:22	a–
math-2326-t1-q5-4	34,195	09/12/2011	00:23	a–
template	37,548	07/21/2011	19:24	a–

[..]		<DIR>	07/30/2011	10:45	—
test-0-5000	docx	3,182,160	07/29/2011	23:35	a–
test-100000-105000	docx	3,183,466	07/30/2011	00:23	a–
test-10000-15000	docx	3,183,239	07/29/2011	23:40	a–
test-105000-110000	docx	3,175,851	07/30/2011	00:26	a–
test-110000-115000	docx	3,181,359	07/30/2011	00:28	a–
test-115000-120000	docx	3,184,515	07/30/2011	00:31	a–
test-120000-125000	docx	3,182,228	07/30/2011	00:33	a–
test-125000-130000	docx	3,179,647	07/30/2011	00:36	a–
test-130000-135000	docx	3,179,880	07/30/2011	00:38	a–
test-135000-140000	docx	3,177,462	07/30/2011	00:41	a–
test-140000-145000	docx	3,174,780	07/30/2011	00:43	a–
test-145000-150000	docx	3,179,052	07/30/2011	00:46	a–
test-150000-155000	docx	3,182,749	07/30/2011	00:48	a–
test-15000-20000	docx	3,179,026	07/29/2011	23:42	a–
test-155000-160000	docx	3,186,131	07/30/2011	00:51	a–
test-160000-165000	docx	3,179,716	07/30/2011	00:53	a–
test-165000-170000	docx	3,176,107	07/30/2011	00:56	a–
test-170000-175000	docx	3,178,196	07/30/2011	00:59	a–
test-175000-180000	docx	3,180,116	07/30/2011	01:01	a–
test-180000-185000	docx	3,176,812	07/30/2011	01:03	a–
test-185000-190000	docx	3,177,382	07/30/2011	01:06	a–
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test-200000-205000	docx	3,172,996	07/30/2011	01:13	a–
test-20000-25000	docx	3,181,452	07/29/2011	23:44	a–
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test-220000-225000	docx	3,181,219	07/30/2011	01:23	a–
test-225000-230000	docx	3,185,913	07/30/2011	01:26	a–
test-230000-235000	docx	3,184,646	07/30/2011	01:28	a–
test-235000-240000	docx	3,176,807	07/30/2011	01:30	a–
test-240000-245000	docx	3,172,987	07/30/2011	01:33	a–
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test-25000-30000	docx	3,182,087	07/29/2011	23:47	a–
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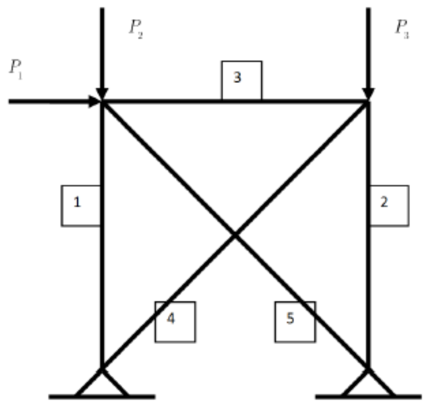
test-275000-280000	docx	3,177,966	07/30/2011	01:49	a–

Automated reports in Latex



200 pages

Table 260: Interval data (uncertainty 37%)

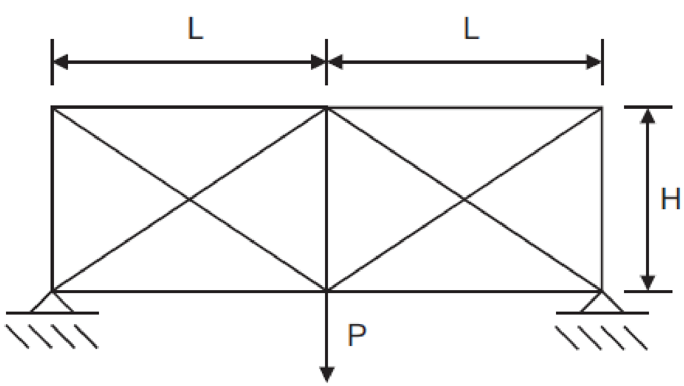
Name	Value	Units
E	[126000000000, 274000000000]	$[\frac{N}{m^2}]$
A	0.0001	$[m^2]$
P	[630, 1370]	$[N]$
L	1	$[m]$
H	1	$[m]$

Table 261: Interval displacements (uncertainty 37%).

u	$\underline{u}[m]$	$mid(\mathbf{u})[m]$	$\overline{u}[m]$
1	1.21305600352052E-05	0.000100990965484165	0.000252372472465261
2	-6.85996385258009E-05	-1.05605747310197E-05	4.59064536921089E-05
3	1.3919764776875E-05	9.04303907531449E-05	0.000230202461510371
4	-0.000147816134050454	-6.05605747310197E-05	-2.12951211825158E-05

Table 262: Combinations (uncertainty 37%, -1 - lower bound, 1 - upper bound, 0 - mid point solution).

u	E_1	E_2	E_3	E_4	E_5	P_1	P_2	P_3
1 (inf)	-1	1	-1	-1	1	-1	1	-1
1 (sup)	-1	-1	-1	-1	-1	1	-1	1
monotone	no	yes	no	no	yes	yes	yes	yes
2 (inf)	-1	1	1	1	-1	-1	1	-1
2 (sup)	-1	-1	-1	-1	1	1	-1	1
monotone	no	yes	yes	yes	yes	yes	yes	yes
3 (inf)	-1	1	1	-1	1	-1	1	-1
3 (sup)	-1	-1	1	-1	-1	1	-1	1
monotone	no	yes	no	no	yes	yes	yes	yes
4 (inf)	-1	-1	1	1	-1	1	-1	1
4 (sup)	-1	1	1	1	1	-1	1	-1
monotone	no	yes	no	no	yes	yes	yes	yes



550 pages

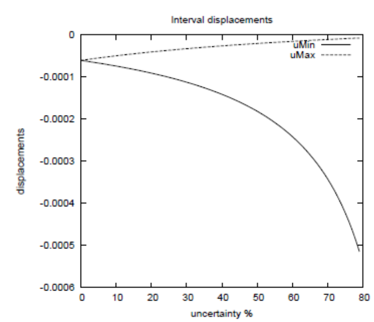


Figure 4: Uncertainty in displacements: u_2

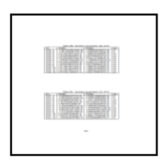


Plate equation

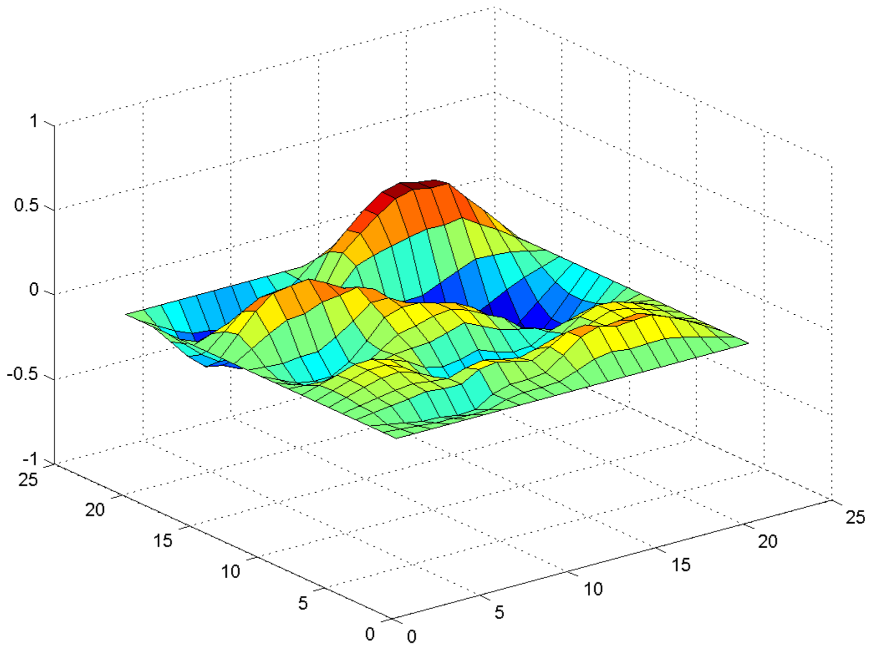
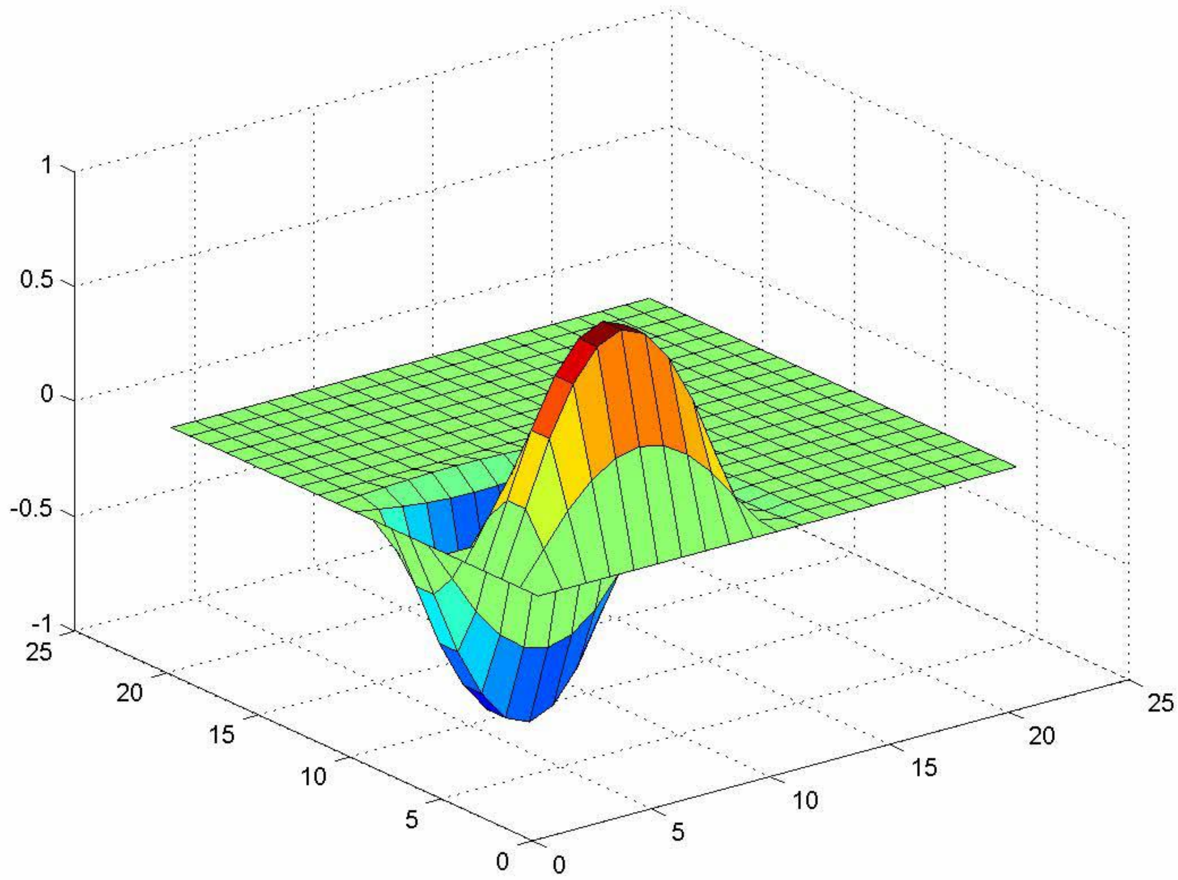


Plate equation



Thank you

I will be back …
with new results soon