Towards a Formal Taxonomy of Hybrid Uncertainty Representations

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Abstract

In this paper we present some preliminary ideas about how to formally relate various uncertainty representations together in a taxonomic structure, capturing both syntactic and semantic generalization. Fuzziness and nonspecificity are presumed as primitive concepts of uncertainty, and transitive and intransitive methods operating with nonspecificity and fuzziness are introduced to generate a base class of hybrid uncertainty representational forms. Additive, maximal, and interval constraints then complete the characterization of the most important hybrid forms.

1 Introduction

Recent years have seen a proliferation of methods in addition to probability theory to represent information and uncertainty, including fuzzy sets and systems [14], fuzzy measures [20], rough sets [15], random sets [11] (Dempster-Shafer bodies of evidence [6]), possibility distributions [2], imprecise probabilities [19], etc. We can identify these fields collectively as General Information Theory (GIT) [12]. So it is clear that there is a pressing need for the GIT community to synthesize these methods, searching out larger formal frameworks within which to place these various components with respect to each other. And indeed there is a growing movement in that direction [4, 13].

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There has also been significant work to develop the semantic relations among these components of GIT [18]. Each was originally intended to capture different semantic aspects of uncertainty and information. Traditionally, these semantic criteria include such categories as fuzziness, vagueness, nonspecificity, conflict, imprecision, belief, plausibility, and randomness. Fig. 1, adapted from Klir [13], shows a typical example of the understood relations among broad classes of uncertainty types.

The authors have been doing significant work with *hybrid* mathematical structures which, it would seem, therefore capture or represent multiple form of uncertainty. In particular we have been working with random sets [9] and intervals [10], which combine nonspecificity and randomness; and with evidence sets [16, 17], which introduce fuzziness weighted by a random set. We therefore have an interest in how these hybrid mathematical forms can be incorporated both within a larger mathematical framework, and within the understanding of the interactions among these semantic categories.

In this paper we present some preliminary ideas about how to develop a simple and exhaustive formal taxonomic basis for the representation of hybrid uncertainty forms sufficient to accommodate these particular structures, and other hybrid and complex uncertainty representation forms. We will do so by making the mathematical development of the formalism interactive with their semiotic basis in different semantic interpretations. In this way we aim not only to achieve greater mathematical elegance and generalization, but also a greater semantic coherence among many possible interpretations.

2 The Semiotics of Uncertainty Representations

Our basic assumption, derived from the semiotic perspective on formal language [5], is that mathematical systems are fundamentally independent of their interpretations. That is, we are free to interpret the symbols and productions of a mathematical system in any way we choose, constrained only by the internal logic and consistency of the formal system itself.

Of course, this does not require that formalisms and their interpretations do not interact at all. First, real formalisms are almost all developed within a specific semantic context, which is surely not harmful. Then, although different formalisms, with different axiomatic bases, do not *require* a specific interpretation, they do provide different *abilities* to represent natural language concepts or scientific applications.

Furthemore, in general a formalism might have *multiple* possibile interpretations, each of which is "valid" for those people who find a benefit in using it in that way. Similarly, it might be possible, and even desirable, to represent a certain semantic context in more than one formal system. Ideally, syntactic (mathematical) generalization can both aid and be aided by the semantic analysis available in terms of the conceptual categories outlined above.

Thus we arrive at a picture illustrated in Fig. 2. At the syntactic level, various mathematical systems have formal entailments among them, as indicated by the dashed arrows. Each may also have multiple semantic interpretations, as indicated by the solid arrows, and vice versa. What is demmanded is that the mathemtical and semantic development go on in the context of each other. So, for example, if a mathematical system has a particular interpretation, and at the same time a formal relation to another mathematical system, then we should attempt to interpret that second system in the original semantic context.

A specific example using the classical structures for uncertainty representation in GIT is shown in figure Fig. 3. Fuzzy sets are almost always interpreted as linguistic variables modeling the vagueness of human language, and probability distributions as a constraint on likelihood or frequency of occurence. Similarly, frequently possibility distributions are interpreted as a form of "elastic constraint" or a graded nondeterminism, and simple intervals as results of imprecise observations or measurements.

But formally, we know that both probability distributions and possibility distributions are specialized fuzzy sets, and intervals are specialized possibility distributions. It is thus incumbent on us to at least *consider*, for example, interpreting a probability distribution as a linguistic variable, or an additive fuzzy set as a statement of likelihood.

3 Mathematical Definitions

Assume a simple finite universe of discourse $\Omega = \{\omega_i\}, 1 \le i \le n$. Denote a subset $A \subseteq \Omega$, and a class $\mathcal{C} = \{A\} \subseteq 2^\Omega$. A **fuzzy subset** denoted $\widetilde{A} \mathrel{\widetilde{\subseteq}} \Omega$ is defined by its membership function $\mu_{\widetilde{A}} \colon \Omega \mapsto [0,1]$, also denoted as just $\widetilde{A}$. The value $\widetilde{A}(\omega_i)$ is usually interpreted as the degree or extent to which $\omega_i \in \widetilde{A}$.

Let an **operator** $\oplus$ be a binary function $\oplus \colon \mathbb{R}^2 \mapsto \mathbb{R}$ which is commutative, associative, usually monotonic, and with identify 0. For a given fuzzy set $\widetilde{A}$, if $\bigoplus_{i=1}^n \widetilde{A}(\omega_i) = 1$ then $\widetilde{A}$ is said to be **normal** by $\oplus$, or to be a **distribution** [9]. When $\oplus = +$, so that $\sum_{i=1}^n \widetilde{A}(\omega_i) = 1$, then $\widetilde{A}$ is an additive **probability distribution**. When $\oplus = \vee$, where $\vee$ is the maximum operator, so that $\bigvee_{i=1}^n \widetilde{A}(\omega) = 1$, then $\widetilde{A}$ is a maximal **possibility distribution**.

Denote the set of all fuzzy subsets of Ω as the fuzzy power set $[0,1]^\Omega$. The weights of a **type-2 fuzzy subset** are themselves fuzzy subsets of $[0,1]$, and are thus defined by membership functions of the form $\widetilde{A} \colon \Omega \mapsto [0,1]^{[0,1]}$. A **level-2 fuzzy subset** is a fuzzy subset of the fuzzy power set of Ω, and thus defined by membership functions of the form $\widetilde{A} \colon [0,1]^\Omega \mapsto [0,1]$.

Given a probability space $\langle X, \Sigma, \Pr \rangle$, where $\langle \cdot \rangle$ denotes a vector, then a function $S \colon X \mapsto 2^\Omega - \{\emptyset\}$, where $-$ is set subtraction, is a **random subset** of Ω if S is Pr-measurable, so that $\forall A \subseteq \Omega, S^{-1}(A) \in \Sigma$. Thus a general random set S associates a probability $(\Pr \circ S^{-1})(A)$ to each $A \subseteq \Omega$.

When Ω is finite, then following Dubois and Prade [3], let $m \colon 2^\Omega \mapsto [0,1]$ be an evidence function or basic assignment if $m(\emptyset) = 0$ and $\sum_{A \subseteq \Omega} m(A) = 1$. Then $\mathcal{S} := \{\langle A_j, m_j \rangle\}$ is a random subset of Ω, where $1 \le j \le N := |\mathcal{S}| \le 2^n, m_j := m(A_j)$, and $m_j > 0$. The **focal set** of a random set $\mathcal{S}$ is then defined as $\mathcal{F}(\mathcal{S}) := \{A_j : m_j > 0\}$.

Random sets are formally equivalent to bodies of evidence in the Dempster-Shafer theory of evidence [6], where now emphasis is placed on the belief and plausibility fuzzy measures which are determined by the basic assignment (and vice versa). For our purposes, we can regard a random set $\mathcal{S}$ as a fuzzy subset of 2^Ω, so that $\mathcal{S} \colon 2^\Omega \mapsto [0,1]$ and $\mathcal{S} = m$, with the additional additivity requirement that $\sum_{A \in 2^\Omega} \mathcal{S}(A) = 1$.

Given a class $\mathcal{C} \subseteq 2^\Omega$, denote $\mathcal{R}(\mathcal{C}) := \{\mathcal{S} : \mathcal{F}(\mathcal{S}) \subseteq \mathcal{C}\}$ as the set of all random subsets of Ω whose focal sets are all in $\mathcal{C}$. Denote

$$\begin{aligned} \mathcal{D} &:= \{I := [I_l, I_u] : I_l, I_u \in \mathbb{R}, I_l \leq I_u\}, \\ \mathcal{D}(I) &:= \{I' \in \mathcal{D} : I_l \leq I'_l \leq I'_u \leq I_u\}, \qquad I \in \mathcal{D} \end{aligned}$$

as the class of all closed interval subsets of $\mathbb{R}$, and the class of all closed sub-intervals of $I \in \mathcal{D}$, respectively. A **random interval** $\mathcal{A}$ is a member of $\mathcal{R}(\mathcal{D})$ [10]. Note that while $\mathcal{D}(I)$ has nondenumerable cardinality, because Ω is finite the overall structure $\mathcal{A}$ is also finite.

Interval-valued fuzzy subsets are defined by membership functions of the form $\widetilde{A}: \Omega \mapsto \mathcal{D}([0,1])$, so that their membership grades are closed interval subsets of $[0,1]$. Rocha [16, 17] has generalized interval-valued fuzzy sets by introducing structures called **evidence sets**, defined by membership functions of the form $\widetilde{A}: \Omega \mapsto \mathcal{R}(\mathcal{D}([0,1]))$. Thus an evidence set maps each element of Ω to a body of evidence whose focal elements are closed intervals of $[0,1]$, or in other words to a random subinterval of $[0,1]$. Again, because Ω is finite, both interval-valued fuzzy sets and evidence sets are also finite structures.

4 Methodology

We now outline our overall method for developing this taxonomic hierarchy in a number of steps:

1. We begin with both a mathematical and a semantic foundation based on the departure from certainty in two independent directions: one based on **non-specificity** as the fundamental action of forming homogenous collections of multiple possibilities; and the other based on **fuzziness** as the fundamental action of creating a heterogenous collection by weighting a single possible entity.

2. We then proceed to provide two methods for the construction of all the possible hybrid forms for uncertainty representation based on the primitive *syntactic* actions of collecting and weighting. The first method is transitive, generating a lattice of possible forms. The second method is intransitive, generating a

tree of possible forms represented as postfix strings. These methods end up being largely, but not completely, equivalent, as discussed below.

3. The next step is to apply meaningful constraints on the weightings, including additive, maximal, and interval constraints.

4. The final step is to consider the meaningfullness and usefullness of the remaining structures, and eliminate irrelevant forms.

Now we discuss aspects of this method in more detail, referring ahead sometimes to the preliminary results which are shown in Figs. 4–7, and discussed in detail in Sec. 5.

4.1 Foundations in Primitive Uncertainty

Consider that our finite universe of discourse is $\Omega = \{a, b, c\}$. We then want to describe a situation in which we ask a question of the sort "what is the value of a variable x which takes values in Ω?". When there is no uncertainty, or in other words complete certainty, we have a single alternative, say $x = a$. In logical terms, we would say that the proposition $\mathbf{p}$: "the value of x is a" is TRUE.

Our approach begins with two primitive concepts which can change our knowledge of x, each of which represents a different form of uncertainty:

Nonspecificity: First, we can introduce more than one alternative answer to the question, while leaving the *kind* of answer unchanged. Introducing this kind of pure nonspecificity implies that x can be more than one possible a, but still definitely one of those so identified. In other words, we establish a *collection* of possible values.

Mathematically, we have introduced now a subset of possible answers. Logically, we now have a set P of propositions $\mathbf{p}$, one for each of some group of the $\omega \in \Omega$, but the value of each proposition is TRUE. So formally we represent this as a simple homogeneous collection, a subset $A \subseteq \Omega$, for example $A = \{a, b\}$.

Fuzziness: Second, we can introduce more than one possible truth-value as the answer to the question, while still leaving only a single answer. Introducing this kind of pure fuzziness implies that x can be just one a, but to a degree. In other words, we can allow a graduated *weighting* of the identified value.

Mathematically, we have introduced now a numerical weighting $\mu(a) \in [0,1]$ on the answer a. Logically, the proposition $\mathbf{p}$ is still "the value of x is a", but it's truth is now $\mu(\mathbf{p}) \in [0,1]$. So formally we represent this as a simple heterogenous collection, a "fuzzy element" $\langle x, \mu(x) \rangle \in \Omega \times [0,1]$, for example $\langle a, .3 \rangle$.

We assert not only that collections and weighted elements are the paradigmatic forms of primtive uncertainty, but also that the collecting *action* of nonspecificity and the weighting *action* of fuzziness are a sufficient *procedural* foundation for the development of all more comlpex or hybrid forms of uncertainty.

We also assert that this is well justified and motivated both in its mathematical simplicity and its semantic coherence. In other words, we believe that all other more complex forms entail fuzziness, nonspecificity, or both, at both the mathematical and semantic levels, although perhaps with the inclusion of additional constraints which we will introduce below.

One consequence of this view is to reject the idea that probabilistic conflict or possibilistic imprecision are independent categories of uncertainty. Rather, each is a more complex expression of uncertainty involving each of these concepts of fuzziness and nonspecificity.

In other words, a probability distribution in general does not represent the semantic category of conflict in any *pure* sense. Rather, it also entails nonspecificity, in the collection of elements of the universe on which it takes values; and fuzziness, in the various weighting that those values can take, albeit they have an additive constraint present among them.

4.2 Taxonomic Development

In the taxonomic development itself, the intention is to start from the first principle of primitive uncertainty representations and transformations (collecting and weighting), and apply them iteratively to generate the basic framework of all the more complex forms.

We have identified two distinct overall methods:

Transitive: In this approach we consider each entity as either a simple or complex collection, and alternatively apply all possible ways of weighting and collecting to its whole and parts. For example, weighting the parts of a simple collection, yielding a classical fuzzy set, suggests weighting the parts of a weighted element separately. In this way we end up applying transformations of weighting and collecting consistently to all entities seen.

Transitivities exist in the sense of multiple paths of transformations resulting in the same structures. Transformations overall are irreversible, and a lattice structure results. The transitivities in the lattice define equivalence classes of transformations. Some of these will involve an intermediate form. For example, weighting the parts of a subset is equivalent to collecting fuzzy elements, each resulting in a fuzzy set. Or, making a whole collection a weight is equivalent to making the weights of a weighted element a collection. These paths and the intermediate entities are kept distinct.

Other equivalent transformations do not involve an intermediate entity. For example, making the parts of a collection a collection is equivalent to combining separate collections into a new collection. These transformations are identified in the diagram.

Intransitive: In this approach generalizations apply in a strict sequence, so that a generalization is only applied to the portion of its representation which was last generalized. This allows the transformations to be reversible, and introduces a significant amount of order to the space of hybrid representations.

Any uncertainty structure is uniquely defined by the sequence of generalization rules that produce it from the initial no uncertainty situation ($x = a$). Furthermore, this history of generalization constrains the type of uncertainty generalization that can be applied to the present structure, since only the portion of its representation that was just generalized can be generalized again with one of the two production rules.

What results is a tree structure generated by a simple postfix string production system. For the grammar, the atoms are $\{F, N\}$ for fuzziness and nonspecificity, and the productions are:

$$X \mapsto XF$$

$$X \mapsto XN.$$

So using just these two concepts of fuzziness and nonspecificity, with either method we can describe a variety of foundational representations, including a variety of simple sets, weighted elements, and classical fuzzy sets. After more iterations, with other levels of fuzziness or nonspecificity added in, more complex forms such as type 2 fuzzy sets, set-valued fuzzy sets, "fuzzy classes", and "fuzzified classes", appear.

In both methods, each application of a transformation introduces another "level" of either nonspecificity or fuzziness. We can then use the notations $|F|$ and $|N|$ to indicate the number of levels of fuzziness and nonspecificity, respectively. Denote $|NF| := \langle |N|, |F| \rangle$, which is the overall "uncertainty cardinality" of a given structure, however it was generated.

4.3 Syntactic Constraints

Once the overall taxonomic structure has been generated, it is possible to identify structures which have both fuzziness and nonspecificity present, that is where $|NF| \geq \langle 1, 1 \rangle$. Under these conditions, additional constraints can be applied to achieve traditional GIT structures.

Normalization: Fuzzy sets are one of the simplest structures which contain both fuzziness and nonspecificity in that they are a collection of weighted elements. If a normalization operator $\oplus$ is applied to this collection of weights, the resulting structures become appropriately distributed. We identify two of particular significance:

- **Additive:** When $\oplus = +$, then the resulting structure is probabilistically distributed. For example, fuzzy sets become probability distributions, and "fuzzified classes" become random sets (see below).

- **Maximal:** When $\oplus = \vee$, then the resulting structure is possibilistically distributed. For example, fuzzy sets become possibility distributions, and "fuzzified classes" become "possibilistic sets".

Intervals: What we call "set-valued fuzzy elements", or structures with the general form $\langle \omega_i, M(\omega_i) \rangle$, where $M(\omega_i) \subseteq [0,1]$, are generated very early in either the transitive or intransitive taxonomies. These are the simplest structures which contain both fuzziness and nonspecificity in the form of a weight which, conversely to fuzzy sets, is *itself* a collection.

Such structures, and others closely related to them, for example set-valued fuzzy sets, can be constrained so that their weights are intervals subsets of $[0,1]$, resulting, for example, in interval-valued fuzzy elements and interval-valued fuzzy sets.

Universe of Discourse: Other constraints on the base universe of discourse Ω are also available and used somewhat in the results below. For example, we can let $\Omega = \mathbb{R}$ or $\Omega = [0,1]$.

These additional mathematical constraints allow greater semantic expressibility. Specifically, the additivity of probability allows the expression of conflict, or randomness. And the maxitivity of possibility allows the expression of ordinal concepts surrounding distances and capacities [8].

4.4 Semantic Interpretation

Finally, now that these basic structures have been identified, they are available for interpretation. Attempting to semantically identify these various structures results in three possible situations:

- Traditional forms are recovered, for example fuzzy or random sets.
- Nontraditional, but meaningful and suggestive, forms are generated. This is the case, for example, with the structure we call the "fuzzified class", a collection of arbitrarily weighted subsets, whose additively constrained form is a random set.

Both of these first two forms are identified on the diagrams with an appropriate label. But in addition a final case can result:

- Nontraditional, and likely unmeaningful forms are generated.

These are labeled in the diagrams using quotation marks.

5 Results

We now describe some preliminary results of our approach. First is an enumeration of the particular basic and constrained forms which we have examined so far. Then these partial results are detailed in the context of both the transitive and intransitive methods. Results for the transitive method are shown in Figs. 4–5, and those for the intransitive method in Figs. 6–7.

5.1 Basic Structures

In Table 1 we list some basic structures generated, that is, those involving uncertainty and fuzziness in various combinations, with no other constraints. In later sections, we detail the specific development of these forms from within both the transitive and intransitive methods.

The left column indicates $|NF|$. For example, a Type 2 Fuzzy Element has two levels of fuzziness and one level of nonspecificity. For each structure generated, we also indicate a simple canonical example drawn from the universe of discourse $\Omega = \{a, b, c\}$. Finally, we indicate the appropriate label or description of the structure. Some of these labels are novel. Those labels in quotation marks are of questionable semantic significance.

5.2 Constrained Structures

In Table 6 we list some of the constrained forms of particular interest, as described in Sec. 4.3. The base forms from which they are derived, and the form of constraint employed, are also listed.

5.3 Transitive Method

Partial results for the transitive method are shown in Figs. 4 and 5. Fig. 4 shows the base forms through $|N| + |F| \leq 3$, and some of the resluting constrained forms. Fig. 5 continues with some structures with $|N| + |F| \geq 3$, concluding with evidence sets.

In the figures, each particular form is identified by its label and canonical example. Dashed boxes surround forms with identical $|NF|$. Transformations operate on parts or whole, and are so identified. Not all possible transformations are shown, in order to make the complexity of the diagrams reasonable to handle.

Transformations are of three distinct kinds:

Weighting: Solid arrows indicate fuzzy weighting. Weighting transformations can be either to make any of the parts into weights, to give the whole preceding structure a weight, or to *make* the whole preceding structure *itself* a weight of a new element.

Collecting: Dashed arrows indicate nonspecific collecting. Collecting transformations can be either to produce a collection of the preceding form, or to make any of the embedded parts a collection.

Constraint: Dotted arrows indicate forms of constraint, and are appropriately labeled.

Note that the basic transformations of weighting and collecting create higher levels of homogeneous and heterogenous structures. Thus the base forms are essentially multiply-layered hierarchical structures of alternating subsets and vectors.

5.4 Intransitive

Partial results for the intransitive method are shown in Figs. 6 and 7. As above, Fig. 6 shows the base forms through $|N| + |F| \leq 3$, and some of the resluting constrained forms, and Fig. 7 continues with some structures with $|N| + |F| \geq 3$, concluding with evidence sets.

In these figures, the arrows are similar to those in Sec. 5.3, except that types of fuzziness and nonspecificity transformations are not distinguished. Recall that unlike the transitive method which can use its operations in any part of the uncertainty representation of an uncertainty structure, the intransitive method is constrained to applying its methods to only the portion which was last generalized.

For example, fuzzy sets are constructed (uniquely) by applying the fuzziness production rule to a crisp set. This operator associates with each element of the crisp set a weight, thus, to generalize fuzzy sets with this method, we can only apply one of the production rules to these weights. In other words, whatever generalization we pursue, it will always result in some sort of set structure with more and more complicated kinds of weights. In a sense, the sequence of transformation preserves or inherits the primordial structure with $|N, F| = \langle 0, 0 \rangle$.

5.5 Comparison

The transitive method can expand fuzzy sets in more ways, for instance by weighting the whole structure and then collecting them to obtain a level-2 fuzzy set, a structure that is not obtainable with the transitive method.

The intransitive method is thus exploring only subsections of the entire universe of hybrid uncertainty structures which it attempts to simplify. All structures reached by this method are either an extension of a set or of a truth-value. If we start by generalizing the certain situation with the fuzziness transformation, all subsequent uncertainty structures can only be some form of singleton with extended truth-values. If we start with the nonspecificity transformation all structures reached are either a crisp collection of singletons with extended truth-values (e.g. fuzzy sets and its descendants), or nested crisp classes whose elements can be ascribed extended truth-value representations.

This restricted exploration of the space of hybrid uncertainty structures, has the advantage of being reversible, thus defining structures with a unique transformation history. It allows the description of most known, semantically defined, truth-value representations and set structures, including evidence sets.

6 Conclusions and Further Work

In this paper we have attempted to outline the basis for the development of a formal structure for the generation of structures to represent hybrid forms of uncertainty from a valid semiotic basis. The results presented so far are not complete. For example, the full space of all of the transitive transformations through, say, $|N| + |F| \leq 4$ has not been explored.

Further, it will be most interesting to see how other forms of uncertainty representation, such as general fuzzy measures [20], imprecise probabilities [19], and rough sets [15], can be considered from this perspective.

Finally, the approach presented here is somewhat in the spirit of category theory. It will be interested to compare this approach to that of others exploring the categories of fuzzy systems [1, 7].

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Tables

$\|NF\|$	**Example**	**Label**
$1,0$	$\langle a,.3\rangle$	Fuzzy element
$0,1$	$\{a,b\}$	Subset
$2,0$	$\langle a,\langle .3,.3\rangle\rangle$	"Fuzzy weighted element"
	$\langle\langle a,.3\rangle,.3\rangle$	"Weighted fuzzy element"
	$\langle a,\langle .b,.3\rangle\rangle$	"Fuzzy element weighted element"
$0,2$	$\{\{a,b\},\{b,c\}\}$	Class
$1,1$	$\{\langle a,.3\rangle,\langle b,.2\rangle\}$	Fuzzy set
	$\langle a,\{.2.3\}\rangle$	Set-valued fuzzy element
	$\langle\{a,b\},.3\rangle$	"Fuzzified subset"
$1,2$	$\langle a,\{\langle .2,.3\rangle,\langle .2,.5\rangle\}\rangle$	Type 2 Fuzzy Element
	$\{\langle a,\langle .2,.3\rangle\rangle,\langle b,\langle .2,.5\rangle\rangle\}$	"Collection of fuzzy weighted elements"
	$\langle\{\langle a,.2\rangle,\langle b,.3\rangle\},.3\rangle$	"Weighted Fuzzy Set"
$2,1$	$\{\langle a,\{.2,.3\}\rangle,\langle b,\{.3,.4\}\rangle\}$	Set-valued fuzzy set
	$\{\langle\{a,b\},.3\rangle,\langle\{b,c\},.4\rangle\}$	"Fuzzified Class"
	$\{\{\langle a,.3\rangle,\langle b,.2\rangle\},\{\langle b,.2\rangle,\langle c,.3\rangle\}\}$	"Fuzzy Class"
$2,2$	$\{\langle a,\{\langle .3,.2\rangle,\langle .2,.5\rangle\}\rangle,\langle b,\{\langle .4,.2\rangle,\langle .2,.6\rangle\}\rangle\}$	Type 2 Fuzzy set
	$\{\langle\{\langle a,.2\rangle,\langle b,.3\rangle\},.3\rangle,\langle\{\langle b,.3\rangle,\langle c,.4\rangle\},.4\rangle\}$	Level 2 Fuzzy Set

Table 1: Generated unconstrained hybrid uncertainty forms.

Base Form	**Constraint**	**Resulting Form**
Fuzzy set	Additive	Probability distribution
Fuzzy set	Maximal	Possibility distribution
Set-valued fuzzy element	Interval	Interval-valued fuzzy element
Set-valued fuzzy set	Interval	Interval-valued fuzzy set
Fuzzified class	Additive	Random set
Random set	$\Omega=\mathbb{R}$, Interval	Random interval
Random interval	$\Omega=[0,1]$	Random subinterval of $[0,1]$

Table 2: Constrained hybrid uncertainty forms.

Figure Captions

1. A semantic taxonomy of uncertainty types (adapted from Klir [13]).

2. Formalisms and their applications.

3. Some formalisms and their applications in GIT.

4. Partial lattice of uncertainty representations, transitive method, $|N|+|F| \leq 3$.

5. Partial lattice of uncertainty representations, transitive method, $|N|+|F| \geq 3$.

6. Partial tree of uncertainty representations, intransitive method, $|N|+|F| \leq 3$.

7. Partial tree of uncertainty representations, intransitive method, $|N|+|F| \geq 3$.

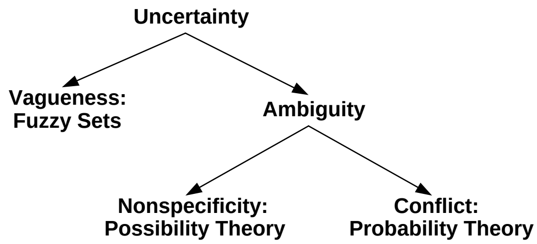

Figure 1: A semantic taxonomy of uncertainty types (adapted from Klir [13]).

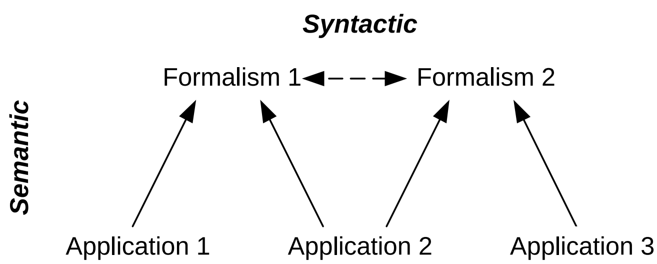

Figure 2: Formalisms and their applications.

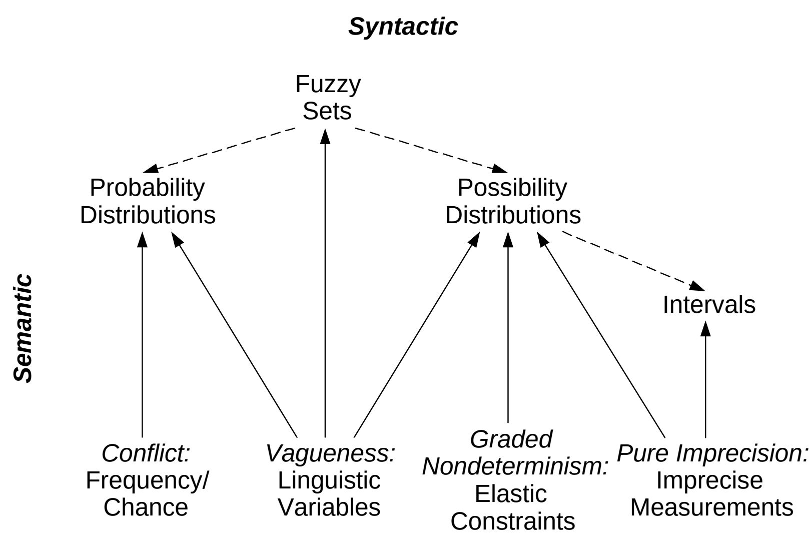

Figure 3: Some formalisms and their applications in GIT.

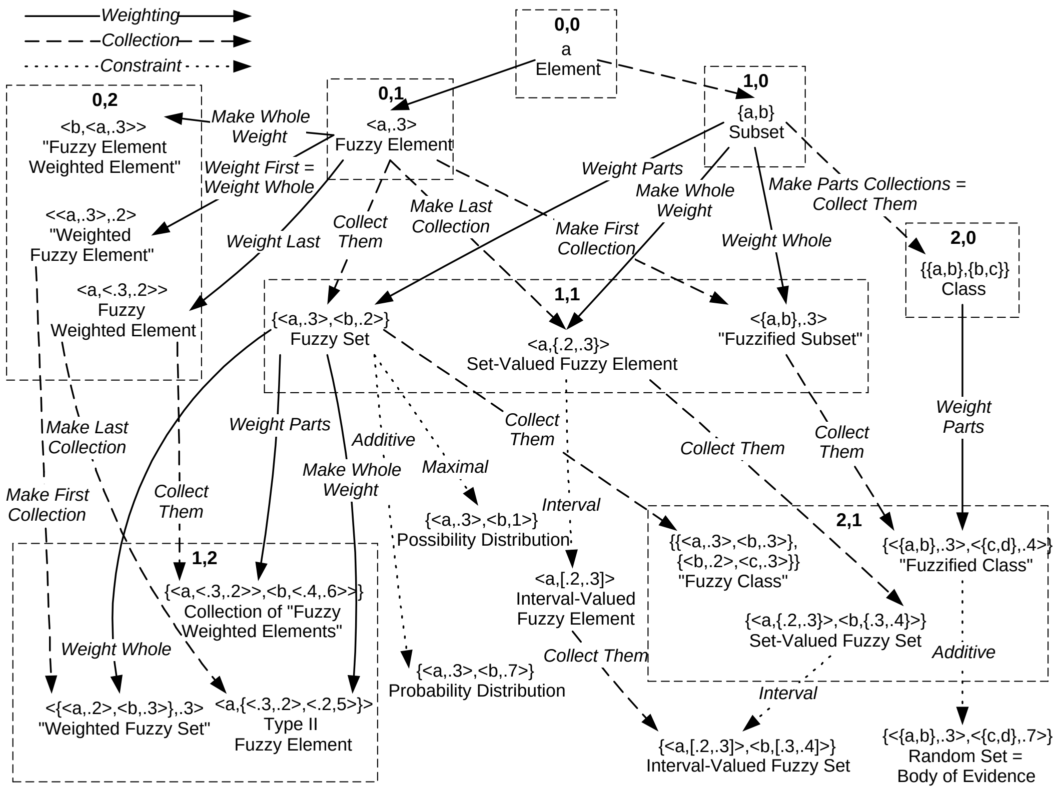

Figure 4: Partial lattice of uncertainty representations, transitive method, $|N| + |F| \leq 3$.

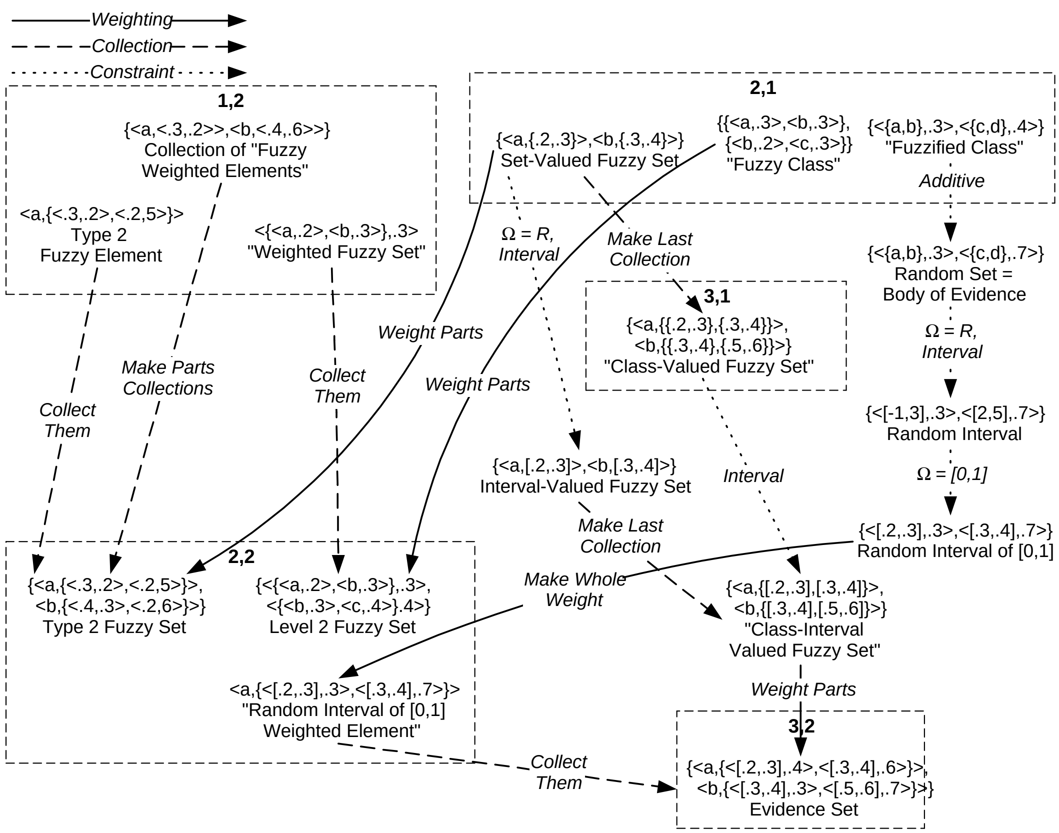

Figure 5: Partial lattice of uncertainty representations, transitive method, $|N| + |F| \geq 3$.

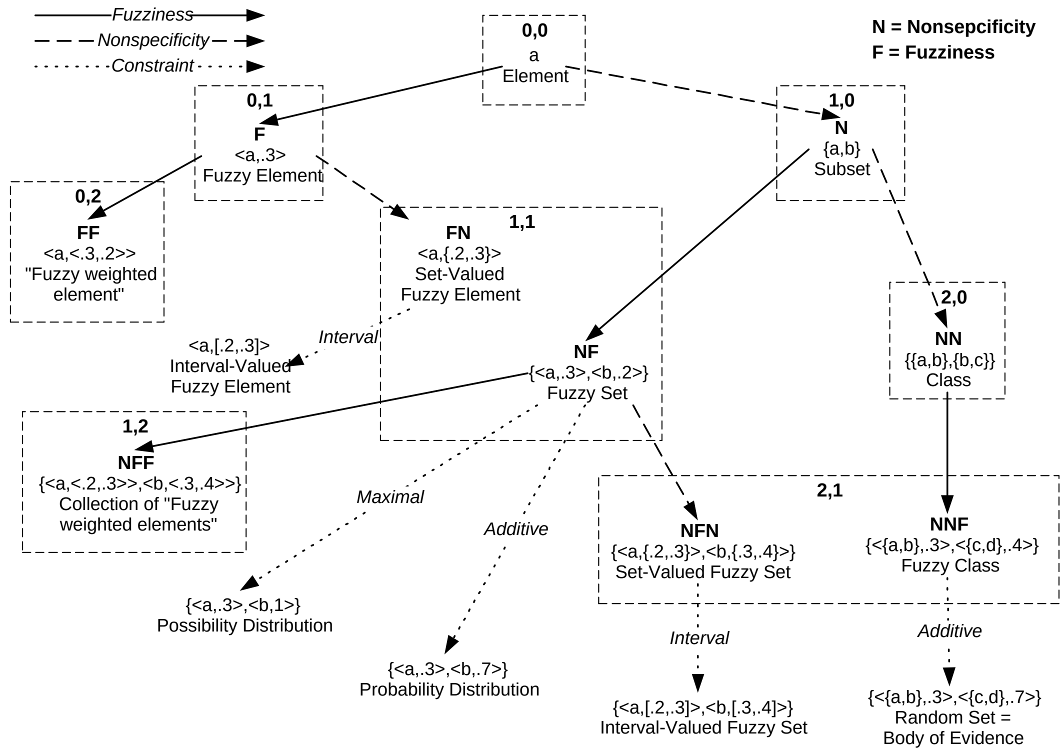

Figure 6: Partial tree of uncertainty representations, intransitive method, $|N|+|F| \leq 3$.

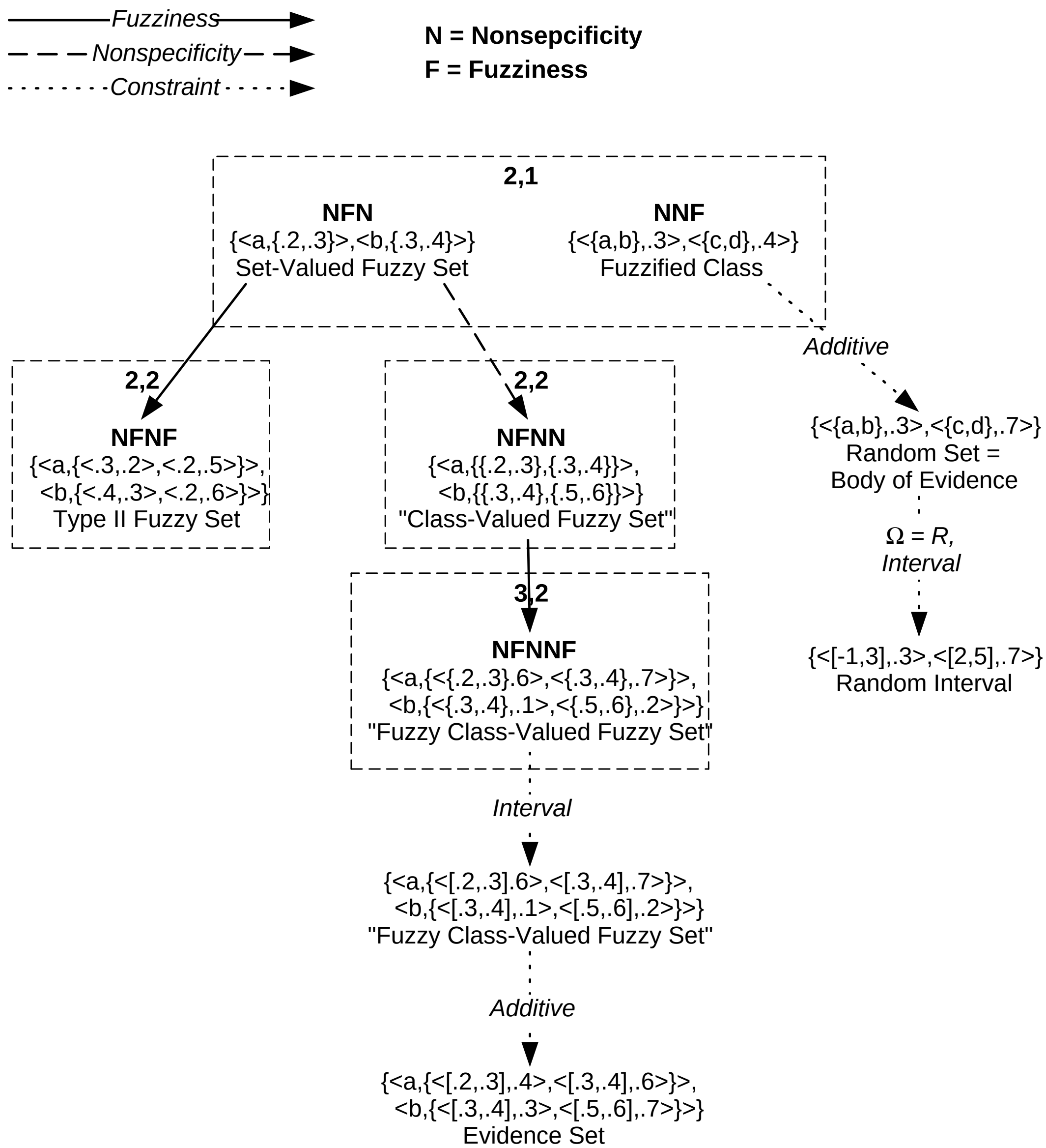

Figure 7: Partial tree of uncertainty representations, intransitive method, $|N|+|F| \geq 3$.