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Calculations of displacement in elastic and elastic-plastic structures with interval parameters

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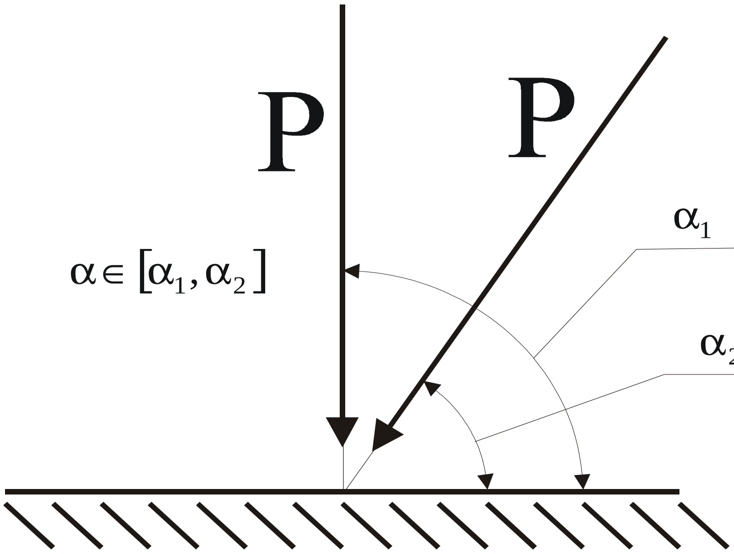
Uncertainties in mechanical systems

Uncertain loads



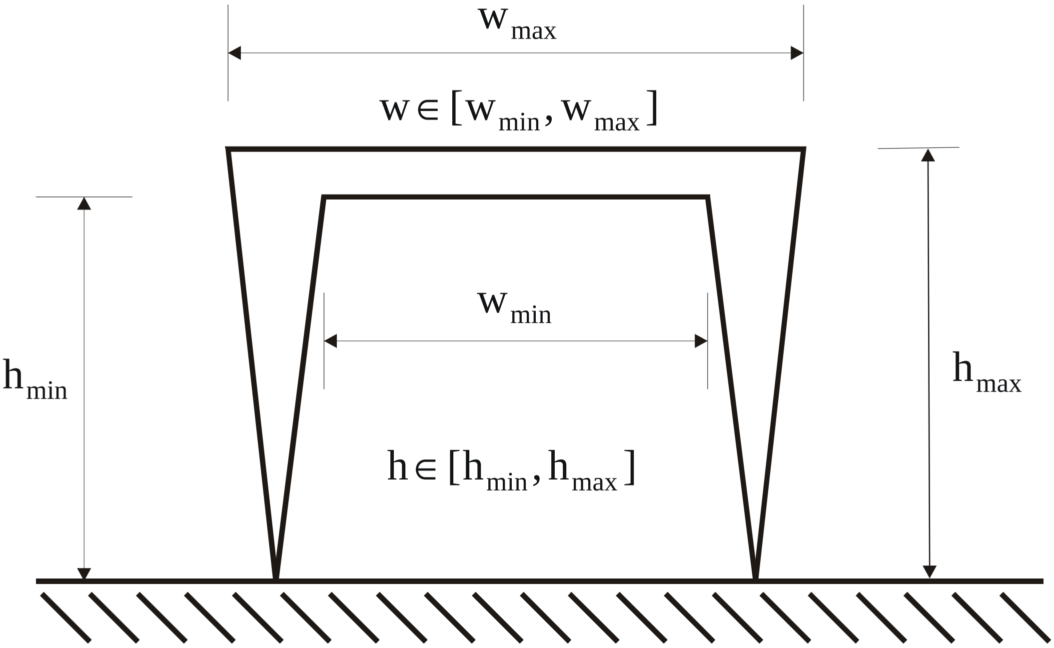
Uncertain material characteristic

$$E \in [E^-, E^+]$$

$$\nu_{xy} \in [\nu_{xy}^-, \nu_{xy}^+]$$

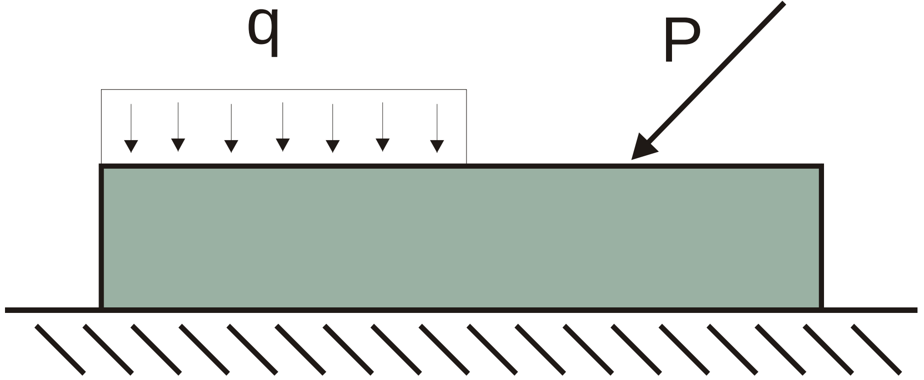
$$G \in [G^-, G^+]$$

Uncertain geometrical parameters



Interval methods

Mechanical system



Finite element method

⇓

Parameter dependent system of linear equations

$$\mathbf{K}(\mathbf{h})\mathbf{q} = \mathbf{Q}(\mathbf{h})$$

Natural interval extension

⇓

System of linear interval equations

$$\hat{\mathbf{K}}([\mathbf{h}])\mathbf{q} = \hat{\mathbf{Q}}([\mathbf{h}])$$

Rohn’s method

Interval Gauss method

⇓

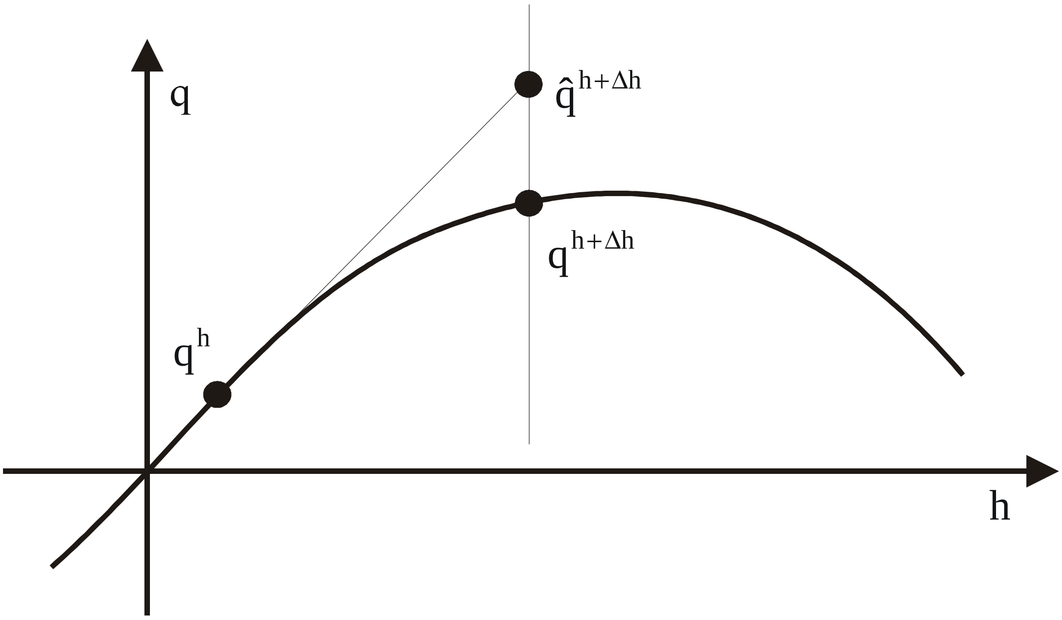
Interval solution

$$\{\mathbf{q} : \mathbf{K}(\mathbf{h})\mathbf{q} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in [\mathbf{h}]\} \subseteq \sum\left(\hat{\mathbf{K}}([\mathbf{h}]), \hat{\mathbf{Q}}([\mathbf{h}])\right)$$

Exact interval methods

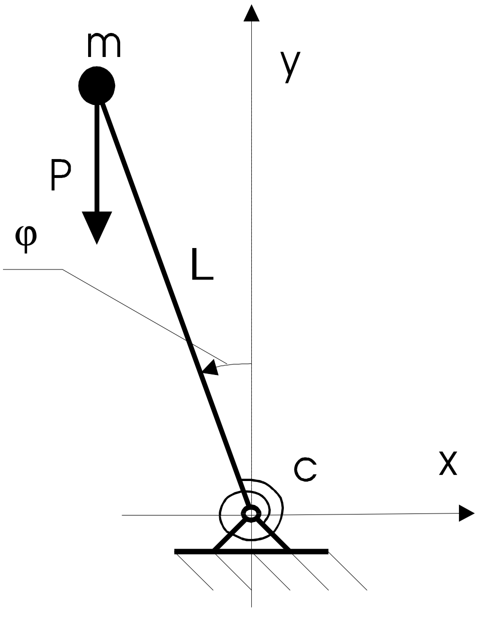
Interval continuation method

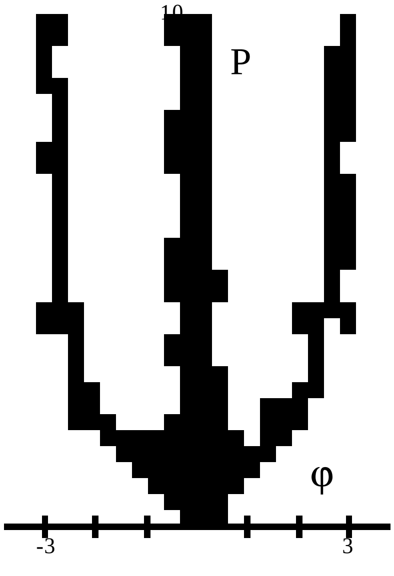
$$\begin{cases} \mathbf{F}(\mathbf{q}, \mathrm{h}) = \mathbf{0} \\ \mathrm{h} - \mathrm{h}^{\tau} = 0 \end{cases}$$



Interval bisection method

$$P \cdot L \cdot \sin(\varphi) - c \cdot \varphi = 0$$





Monotonicity

General definition of the solution set

$$\mathbf{q}([\mathbf{h}]) = \{\mathbf{q} : \mathbf{K}(\mathbf{h})\mathbf{q} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in [\mathbf{h}]\}$$

Extreme values of the solution set

$$q_i^- = \inf\{q_i : (q_1, \ldots, q_m) \in \mathbf{q}([\mathbf{h}])\}$$
$$q_i^+ = \sup\{q_i : (q_1, \ldots, q_m) \in \mathbf{q}([\mathbf{h}])\}$$

Calculation the exact solution
of parameter dependent system of equations
is NP-hard

If the following functions

$$q_i = q_i(\ldots, h_j, \ldots)$$

are **monotone**, then extreme values of the solution set can be calculated using the endpoints of the interval [**h**].

$$q_i^{\pm} = q_i\left(h_1^{\pm}, \ldots, h_m^{\pm}\right)$$

Applications of sensitivity analysis

$$\text{If } \frac{\partial q_i(h_1,\ldots,h_m)}{\partial h_j} > 0, \quad \text{then}$$

$$q_i^- = q_i\left(\ldots,h_j^-,\ldots\right), \qquad q_i^+ = q_i\left(\ldots,h_j^+,\ldots\right)$$

$$\text{If } \frac{\partial q_i(h_1,\ldots,h_m)}{\partial h_j} < 0, \quad \text{then}$$

$$q_i^- = q_i\left(\ldots,h_j^+,\ldots\right), \qquad q_i^+ = q_i\left(\ldots,h_j^-,\ldots\right)$$

We can write it in the following form:

$$q_i^- = q_i\left(\mathbf{h}^{-\mathrm{sign}\left(\frac{\partial \mathbf{q}}{\partial \mathbf{h}}\right)}\right), \qquad q_i^+ = q_i\left(\mathbf{h}^{\mathrm{sign}\left(\frac{\partial \mathbf{q}}{\partial \mathbf{h}}\right)}\right)$$

Interval monotonicity tests

Implicit solutions

It can be shown that if the following interval Jacobian matrices

$$\frac{\partial \hat{\mathbf{F}}([\mathbf{x}],[\mathbf{h}])}{\partial \mathbf{x}} \quad \frac{\partial \hat{\mathbf{F}}([\mathbf{x}],[\mathbf{h}])}{\partial \left(x_1,\ldots,h_j,\ldots,x_n\right)}$$

are regular then solutions of parameter dependent system of equations are monotone.

In order to check regularity of interval matrix interval gauss method Rohn’s method could be applied.

Explicit solutions

If

$$0 \notin \frac{\partial \hat{q}_i([\mathbf{h}])}{\partial h_j}$$

then function $q_i = q_i(\ldots,h_j,\ldots)$ is monotone in the interval $[h_j]$.

Point monotonicity tests

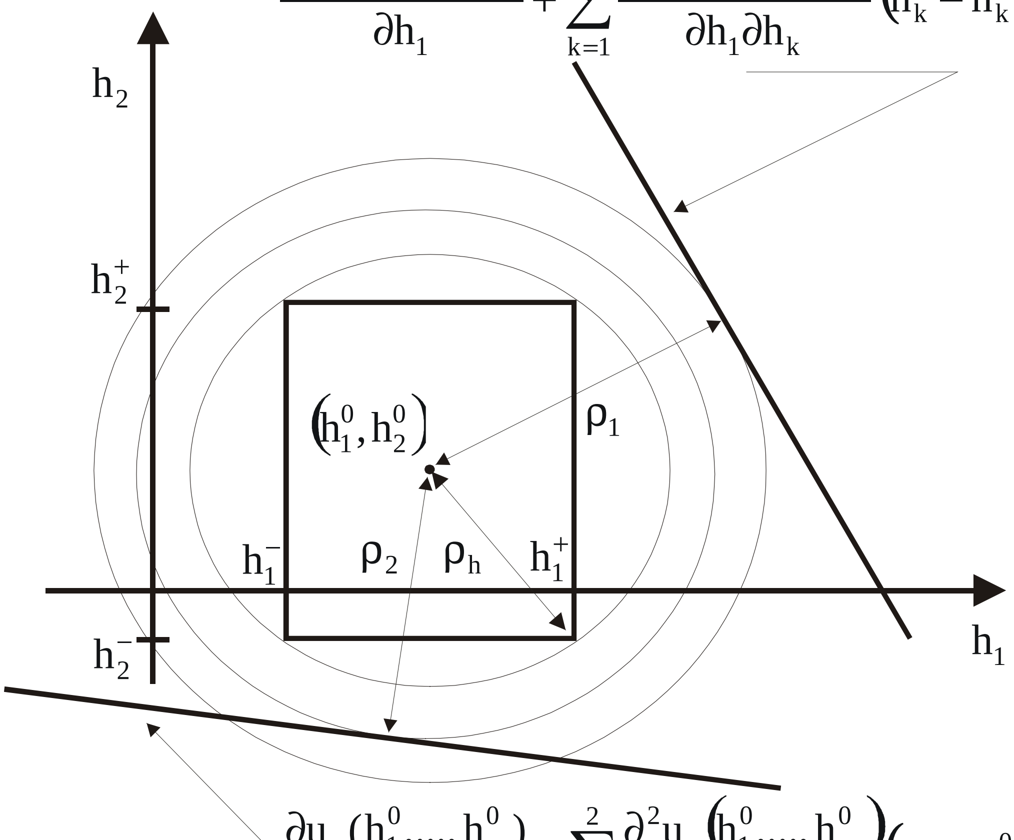
First method

$$\frac{\partial u_x(h_1,\ldots,h_m)}{\partial h_j} \approx \frac{\partial u_x(h_1^0,\ldots,h_m^0)}{\partial h_j} + \sum_{k=1}^{m} \frac{\partial^2 u_x\left(h_1^0,\ldots,h_m^0\right)}{\partial h_j \partial h_k}\left(h_k - h_k^0\right) = 0$$

A distance of these hypersurface from the point $\left(h_1^0,\ldots,h_m^0\right)$ is as follows.

$$\rho_j = \left|\frac{\partial u_x(h_1^0,\ldots,h_m^0)}{\partial h_j}\right| \Bigg/ \sqrt{\sum_{k=1}^{m}\left(\frac{\partial^2 u_x\left(h_1^0,\ldots,h_m^0\right)}{\partial h_j \partial h_k}\right)^2}$$

$$\frac{\partial u_x(h_1^0,\ldots,h_m^0)}{\partial h_1} + \sum_{k=1}^{2} \frac{\partial^2 u_x\left(h_1^0,\ldots,h_m^0\right)}{\partial h_1 \partial h_k}\left(h_k - h_k^0\right) = 0$$



$$\frac{\partial u_x(h_1^0,\ldots,h_m^0)}{\partial h_2} + \sum_{k=1}^{2} \frac{\partial^2 u_x\left(h_1^0,\ldots,h_m^0\right)}{\partial h_2 \partial h_k}\left(h_k - h_k^0\right) = 0$$

Point monotonicity test

Second method

$$\frac{\partial \breve{u}_x(h_1,\dots,h_m)}{\partial h_j} = \frac{\partial u_x(h_1^0,\dots,h_m^0)}{\partial h_j} + \sum_{k=1}^{m} \frac{\partial^2 u_x\left(h_1^0,\dots,h_m^0\right)}{\partial h_j \partial h_k}\left(h_k - h_k^0\right)$$

If

$$\operatorname{sign}\left(\frac{\partial \breve{u}_x\left(h_1^{\pm},\dots,h_m^{\pm}\right)}{\partial h_j}\right) = \text{const}$$

then

$$\left\{(h_1,\dots,h_m) : \frac{\partial \breve{u}_x(h_1,\dots,h_m)}{\partial h_j} = 0\right\} \cap [\mathbf{h}] = \varnothing$$

i.e. $\operatorname{sign}\left(\frac{\partial \breve{u}_x(h_1,\dots,h_m)}{\partial h_j}\right) = \text{const}$ for $h_i \in [h_i]$ i=1,…,m

and function $u_x\left(h_1^0,\dots,h_{j-1}^0,h_j,h_{j+1}^0,\dots,h_m^0\right)$ is monotone in the interval

$[h_j]$ where $h_i^0 \in [h_i]$ i=1,…,m and $i \neq j$.

Point monotonicity test

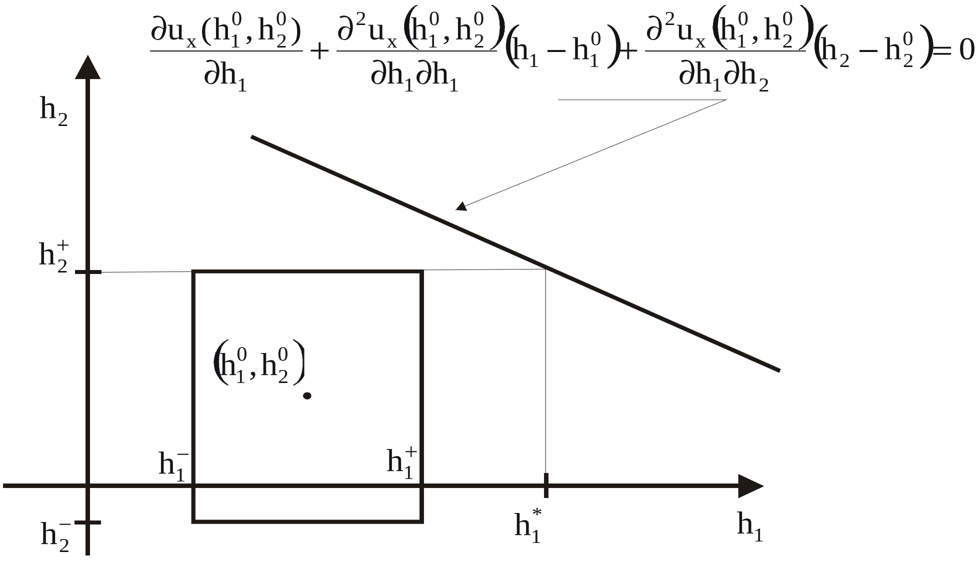
Third method

Let m=2 and $h_2 = h_2^+$

$$\frac{\partial u_x(h_1^0, h_2^0)}{\partial h_1} + \frac{\partial^2 u_x\left(h_1^0, h_2^0\right)}{\partial h_1 \partial h_1}\left(h_1 - h_1^0\right) + \frac{\partial^2 u_x\left(h_1^0, h_2^0\right)}{\partial h_1 \partial h_2}\left(h_2^+ - h_2^0\right) = 0$$

then

$$h_1^* = h_1^0 - \frac{1}{\dfrac{\partial^2 u_x\left(h_1^0, h_2^0\right)}{\partial h_1 \partial h_1}}\left(\frac{\partial u_x(h_1^0, h_2^0)}{\partial h_1} + \frac{\partial^2 u_x\left(h_1^0, h_2^0\right)}{\partial h_1 \partial h_2}\left(h_2^+ - h_2^0\right)\right)$$



$$\frac{\partial u_x(h_1^0, h_2^0)}{\partial h_1} + \frac{\partial^2 u_x\left(h_1^0, h_2^0\right)}{\partial h_1 \partial h_1}\left(h_1 - h_1^0\right) + \frac{\partial^2 u_x\left(h_1^0, h_2^0\right)}{\partial h_1 \partial h_2}\left(h_2 - h_2^0\right) = 0$$

In multidimensional case:

$$h_i^* = h_i^0 - \frac{1}{\dfrac{\partial^2 u_x\left(h_1^0, \ldots, h_m^0\right)}{\partial h_j \partial h_i}}\left(\frac{\partial u_x(h_1^0, h_2^0)}{\partial h_j} + \sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{\partial^2 u_x\left(h_1^0, \ldots, h_m^0\right)}{\partial h_j \partial h_k}\left(h_k^\pm - h_k^0\right)\right)$$

If

$$h_i^* \notin [h_i]$$

then function $u_x\left(h_1^0, \ldots, h_{j-1}^0, h_j, h_{j+1}^0, \ldots, h_m^0\right)$ is monotone in the interval $[h_j]$ where $h_i^0 \in [h_i]$ i=1,…,m and $i \neq j$.

Point monotonicity test

Forth method

$$\frac{\partial u_x\left(h_j\right)}{\partial h_j} := \frac{\partial}{\partial h_j} u_x\left(h_1^0,\ldots,h_{j-1}^0,h_j,h_{j-1}^0,\ldots,h_m^0\right)$$

$$\frac{\partial u_x(h_j)}{\partial h_j} \approx \frac{\partial u_x(h_j^0)}{\partial h_j} + \frac{\partial^2 u_x\left(h_j^0\right)}{\partial h_j^2}\left(h_j - h_j^0\right) = 0$$

$$h_j = h_j^0 - \frac{\dfrac{\partial u_x(h_j^0)}{\partial h_j}}{\dfrac{\partial^2 u_x\left(h_j^0\right)}{\partial h_j^2}}$$

$$\frac{\partial u_x(h_j^0)}{\partial h_j} \approx \frac{u_x(h_j^0 + \Delta h_j^0) - u_x(h_j^0)}{\Delta h_j^0}$$

$$\frac{\partial^2 u_x(h_j^0)}{\partial h_j^2} \approx \frac{u_x(h_j^0 + \Delta h_j^0) - 2 \cdot u_x(h_j^0) + u_x(h_j^0 - \Delta h_j^0)}{\left(\Delta h_j^0\right)^2}$$

$$h_j^* \approx h_j^0 - \frac{\left[u_x(h_j^0 + \Delta h_j^0) - u_x(h_j^0)\right] \cdot \Delta h_j^0}{u_x(h_j^0 + \Delta h_j^0) - 2 \cdot u_x(h_j^0) + u_x(h_j^0 - \Delta h_j^0)}$$

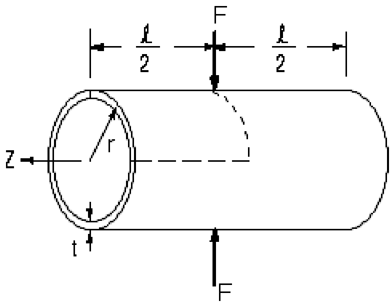
$$h_j^* \notin \left[h_j^-, h_j^+\right] = \left[h_j\right]$$

Thin-walled cylinder

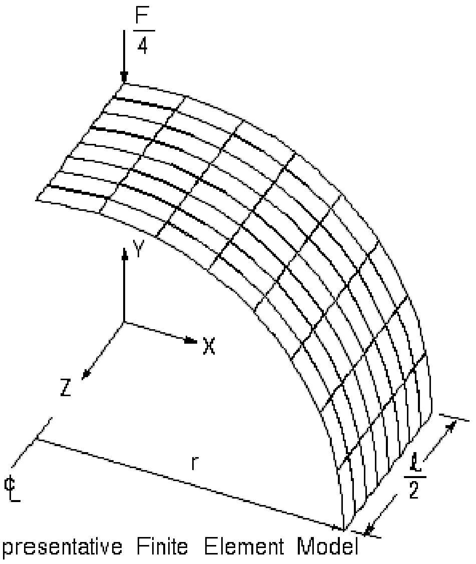
Data

$E \in [2.0 \cdot 10^5, 2.2 \cdot 10^5]$ MPa, $\nu \in [0.2, 0.3]$

L=0.263 m, r=0.126 m, t=$2.38 \cdot 10^{-3}$ m, F=444.8 N



Problem Sketch



Representative Finite Element Model

$$\Delta u_y \overset{df}{=} u_y(h + \Delta h) - u_y(h)$$

$$\Delta^2 u_y \overset{df}{=} u_y(h + \Delta h) - 2 \cdot u_y(h) + u_y(h - \Delta h)$$

$$\Delta h^* \overset{df}{=} \frac{[u_y(h + \Delta h) - u_y(h)] \cdot \Delta h}{u_y(h + \Delta h) - 2 \cdot u_y(h) + u_y(h - \Delta h)}$$

Because

$$2 \cdot 10^{10} \text{ MPa} = E^+ - E^- << \left|\Delta h_E^*\right| = 9.972 \cdot 10^4 \text{ MPa}$$

$$0.1 = \nu^+ - \nu^- << \left|\Delta h_\nu^*\right| = 2.662$$

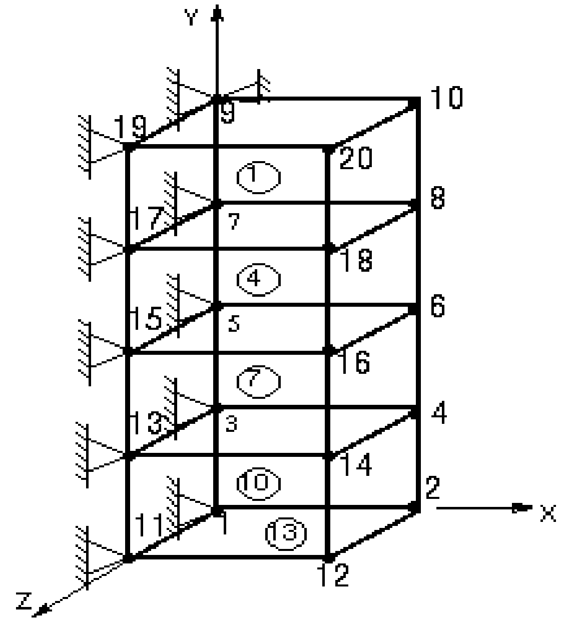
then functions $u_y(E), u_y(\nu)$ are monotone in the intervals [E],[ν].

The interval solution $u_y \in [-0.043514, -0.03748]$.

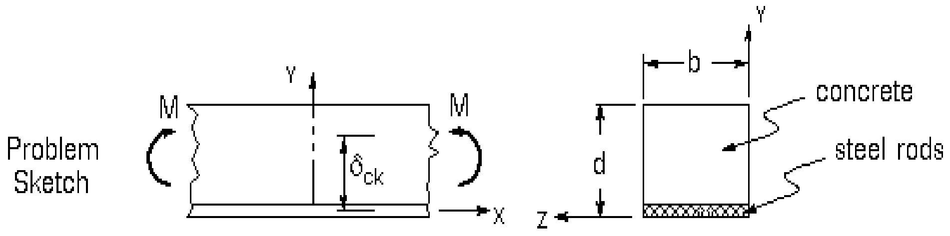
Reinforced Concrete Beam

Data

Concrete	Steel	Geometry
$E \in [1.3,1.5] \cdot 10^4$ MPa	$E \in [2.0,2.2] \cdot 10^5$ MPa	$a = 0.127$ m
$\sigma_{ct} = 0$ MPa	$\nu \in [0.2,0.3]$	$b = 0.152$ m
$\nu = 0$		$A = 0.019\ \text{m}^2$



Representative Finite Element Model



The interval solution $u_{2x} \in [0.182, 0.200] \cdot 10^{-4}$ [m].

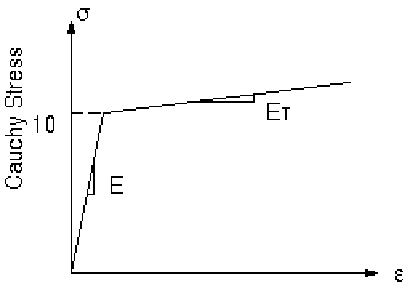
Large strain in-plane torsion

Data

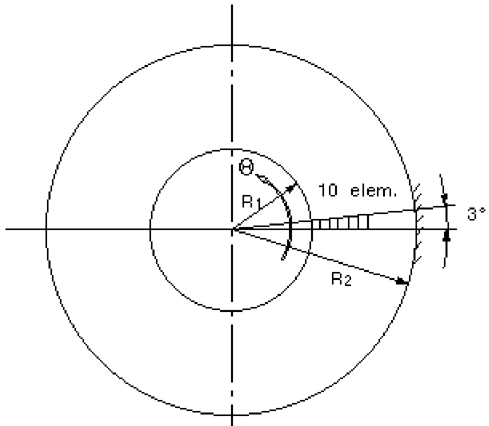
$E \in [49.3,\ 49.9]$ MPa $\qquad E_T \in 0.28\,\text{MPa} \qquad \sigma_y = 0.6\,\text{MPa}$

$\nu \in [0.32,\ 0.34] \qquad R_1 = 0.254\,\text{m} \qquad R_2 = 0.508\,\text{m}$

$\Theta = 60^o$



Logarithmic Strain
Stress-Strain Curve



Problem Model

We assume that

$$h_1 = E, \quad h_2 = \nu$$

Because

$$h_i^* \notin \left[h_i^-, h_i^+\right] \quad \text{for} \quad i = 1,2$$

then the functions

$$\sigma_x = \sigma_x(h_1), \quad \sigma_x = \sigma_x(h_2)$$

are monotone in the intervals $[h_1],[h_2]$.

Interval solution

$$\sigma_x \in [0.36,\ 0.48]\,\text{MPa}$$

Fuzzy parameters

Measurements

$$\Downarrow$$

Random set (Ξ, m)

$$\Xi = \{[x_1], \ldots, [x_n]\}, \qquad m([x_i]) = \frac{1}{n}$$

$$[x_i] = [x_i^-, x_i^+] \subset R$$

$$\Downarrow$$

$$\Xi = \bigcup_i \Xi_i, \quad \bigcap_j [x_j] \neq \varnothing, \quad [x_j] \in \Xi_i$$

$$\Downarrow$$

$$\mu^i(x) = \sum_{x \in [x_j] \wedge [x_j] \in \Xi_i} m([x_j])$$

$$\Downarrow$$

$$Pl(A) = \sum_i \sup_{x \in A} \mu^i(A)$$

$$\Downarrow$$

$$Pl(y) = \sum_i \sup_{y = f(x_1, \ldots, x_n)} \mu^i(x_1) \wedge \ldots \wedge \mu^i(x_n)$$

$$\Downarrow$$

$$Pl(A) = \sum_i \sup_{\substack{y = f(x_1, \ldots, x_n) \\ y \in A}} \mu^i(x_1) \wedge \ldots \wedge \mu^i(x_n)$$

Application of sensitivity analysis to calculation of structures with uncertain parameters

Random set (Ξ, m)

$$\Downarrow$$

$$\mu^{i}(x) = \sum_{x\in[x_j]\wedge[x_j]\in\Xi_i} m([x_j])$$

$$\Downarrow$$

α-level-cut

$$x^{i}_{\alpha} = \{x : \mu(x) \geq \alpha\}$$

$$\Downarrow$$

applications

$$\boxed{q^{i}_{\alpha} = \{q(x) : x \in x^{i}_{\alpha}\}}$$

$$\Downarrow$$

$$\mu^{i}_{q}(q) = \sup\{\alpha : q \in q^{i}_{\alpha}\}$$

$$\Downarrow$$

$$\mathrm{Pl}(q) = \sum_{i} \mu^{i}(q)$$

$$\Downarrow$$

$$[q] = \bigcup_{i} q^{i}_{0}$$

$$\mathrm{Pl}([q]) = \sum_{i} \sup\{\mu^{i}(q) : q \in [q]\}$$

Conclusion

1) **Presented method are a very effective tools for modelling of structures with uncertain parameters.**

2) **Presented method can be applied when intervals** $[h_j]$ **are sufficiently narrow.**

3) **Presented algorithms can be extend to the case when uncertain parameters will be fuzzy numbers.**

4) **These methods can be applied to nonlinear problems of computational mechanics.**

5) **Presented algorithms use results generated by an existing engineering software.**

6) **Presented methods are universally applicable.**

7) **When uncertain parameters are fuzzy numbers presented method can be applied to calculation upper and lower probability of the safety of structures.**

8) **When intervals are to wide the interval monotonicity test or the Monte-Carlo simulations should be applied. Unfortunately these methods have a higher computational complexity.**