Worst-Case Analysis, Safety Margins, and Fuzzy Algebra: A Mathematical Equivalence

With Applications to Truss Structures and Multivalued Logic

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Abstract

This report establishes a rigorous mathematical equivalence between classical structural safety analysis based on allowable stresses and safety factors, fuzzy set theory and its α-cut formalism, and many-valued logic.

The central result is that the safety margin $\mathrm{SM}_i(P_1, P_2)$ of every structural member naturally defines a fuzzy membership function on the load space, mapping every load combination into $[0, 1]$. The α-cut condition $\mathrm{SM}_i \geq \alpha$ recovers exactly the admissible load domain corresponding to the safety-factor threshold $\mathrm{SF}_{\mathrm{req}} = 1/(1 - \alpha)$. For nonredundant (series-system) structures, global worst-case analysis is governed by

$$\mathrm{SM}(P_1, P_2) = \min\bigl\{\mathrm{SM}_1(P_1, P_2), \ldots, \mathrm{SM}_m(P_1, P_2)\bigr\},$$

which is the standard fuzzy-intersection formula (Godel/minimum t-norm).

A second layer of algebra appears *within a single bar* when one separates the effect of two loads and compares the resulting one-load margins $\mathrm{SM}_{5,1} = \mathrm{SM}_5(P_1, 0)$ and $\mathrm{SM}_{5,2} = \mathrm{SM}_5(0, P_2)$ with the combined margin $\mathrm{SM}_5(P_1, P_2)$. For bar 5 of the five-bar truss, the exact composition law is sign-dependent: co-directional loading yields the strong Łukasiewicz conjunction, whereas opposite-sign loading yields the Łukasiewicz equivalence. The associated residuum has a direct interpretation as residual capacity. The report also explains why the classical min-based extension principle does not reproduce the exact safety profile of bar 5 on load space, whereas a Łukasiewicz T_L-extension does. On fixed sign sectors this exact load-space description agrees with the constant-sign formulas obtained directly in the algebra of safety margins. The framework is illustrated on a five-bar plane truss.

Contents

1 Introduction

Engineering design uses deterministic safety criteria every day. A structural element is deemed acceptable if its internal stress remains below a prescribed allowable value; a safety factor specifies by how much the capacity exceeds the demand. These ideas are centuries old and are codified in building codes worldwide [17, 18].

Fuzzy set theory, introduced by Zadeh [1], provides a mathematical language for gradual transitions: an element does not simply pass or fail — it belongs to the "safe" set to some *degree*. The extension principle [2, 3] allows fuzzy quantities to be propagated through arbitrary functional relationships.

At the same time, the classical many-valued logics of Łukasiewicz [7] and Godel replace the Boolean truth values $\{0, 1\}$ with the unit interval $[0, 1]$, interpreting intermediate values as partial or graded truth [10, 11].

The purpose of this report is to show that these three threads — structural safety, fuzzy algebra, and multivalued logic — are *mathematically equivalent* in the context of nonredundant load-carrying structures. No probabilistic or possibilistic assumptions are required; the equivalence is a purely algebraic consequence of the safety margin definition.

A short preliminary version of the main equivalence was presented in [15]. The present report develops the full five-bar truss example, the exact constant-sign formulas for bar 5, and the comparison between the classical min-based extension principle and the exact Łukasiewicz T_L-extension on load space. It also fits into a broader line of research on combining and interpreting different mathematical descriptions of uncertainty and gradation in engineering [16].

Two distinct aggregation levels must be separated. Across different members of a nonredundant structure, safety is governed by the weakest-link minimum and therefore by the Godel conjunction. Within a *single* member subjected to several loads, however, one may compare the margin produced by each load acting alone with the margin produced by the actual superposed load. For bar 5 of the truss this local superposition algebra is not the global minimum: on constant-sign sectors it induces specific Łukasiewicz connectives on the one-load margins. This distinction between *across-bar aggregation* and *within-bar superposition* is central to the updated report.

Remark (No Expert Elicitation Required)

The membership function in this framework is *derived* from the physics of the structure, not chosen by an expert. Every parameter — cross-sectional area A_i, allowable stress $\sigma_{\max}$, internal force N_i — has a direct engineering meaning. Classical deterministic analysis therefore implicitly encodes a fuzzy model, even when carried out with purely crisp arithmetic.

The development is self-contained. Section 2 defines the safety margin. Section 3 shows that it naturally defines a fuzzy membership function and derives the α-cut equivalence. Section 4 establishes the global fuzzy-intersection formula for series systems. Section 5 works through a five-bar truss. Section 6 relates load propagation to the extension principle. Section 7 analyses the dependency problem. Section 8 identifies the safety margin as a degree of truth. Section 9 develops the exact composition law for bar 5 in terms of the one-load margins $\mathrm{SM}_{5,1}$ and $\mathrm{SM}_{5,2}$, explains why a naive classical extension-principle formulation on margin space is insufficient, and introduces a generalised extension principle for the mechanically induced connectives.

2 Safety Margins and Safety Factors in Structural Mechanics

2.1 The prismatic bar under axial load

Consider a uniform prismatic bar (member i) with cross-section $A_i > 0$ carrying axial internal force N_i (Figure 1). The material has allowable stress $\sigma_{\max} > 0$.

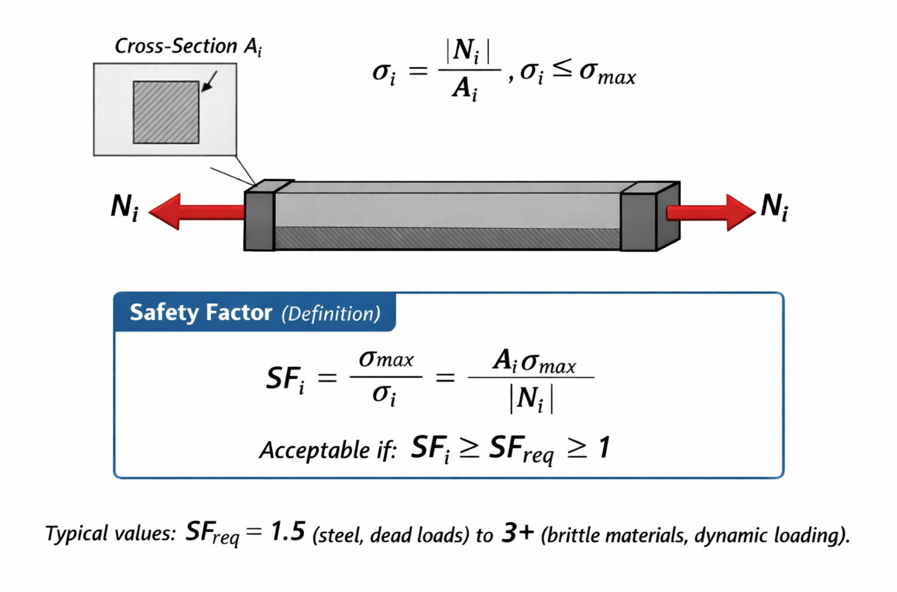


Figure 1: Prismatic bar under axial compression. The applied load P induces an internal force N and stress $\sigma = |N|/A$. The safety factor $\mathrm{SF} = \sigma_{\max}/\sigma$ quantifies the reserve capacity.

The axial stress is $\sigma_i = |N_i|/A_i$, and the classical adequacy criterion is $\sigma_i \le \sigma_{\max}$ [19, 20].

2.2 Safety factor

Definition 2.1 (Safety Factor)

The *safety factor* of member i is

$$\mathrm{SF}_i = \frac{\sigma_{\max}}{\sigma_i} = \frac{A_i\,\sigma_{\max}}{|N_i|}.$$

The element satisfies the design criterion iff $\mathrm{SF}_i \ge \mathrm{SF}_{\mathrm{req}}$ for a specified $\mathrm{SF}_{\mathrm{req}} \ge 1$.

Typical values range from 1.5 (steel, dead loads) to 3 or higher for brittle materials or dynamic loading [18].

2.3 Safety margin

Definition 2.2 (Safety Margin)

The *safety margin* of member i is the normalised reserve capacity

$$\mathrm{SM}_i = \max\Bigl(0,\ 1 - \frac{|N_i|}{A_i\,\sigma_{\max}}\Bigr) = \max\Bigl(0,\ 1 - \frac{\sigma_i}{\sigma_{\max}}\Bigr).$$

By construction $\mathrm{SM}_i \in [0,1]$: $\mathrm{SM}_i = 1$ means the bar is unloaded; $0 < \mathrm{SM}_i < 1$ means it is stressed within the allowable range; $\mathrm{SM}_i = 0$ means the allowable stress is reached or exceeded. Figure 2 shows the ramp shape.

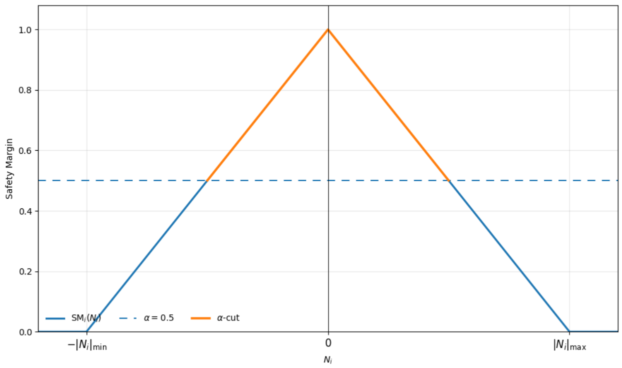


Figure 2: Safety margin $\mathrm{SM}_i = \mathrm{SM}_i(N_i)$: a decreasing ramp from 1 (zero load) to 0 (allowable stress reached). No expert elicitation required; the curve follows directly from $\sigma_{\max}$ and A_i.

2.4 Algebraic relation between SM and SF

A direct calculation gives

$$\mathrm{SF}_i = \frac{1}{1 - \mathrm{SM}_i}, \qquad \mathrm{SM}_i = 1 - \frac{1}{\mathrm{SF}_i}.$$

The design condition $\mathrm{SF}_i \geq \mathrm{SF}_{\mathrm{req}}$ is equivalent to

$$\mathrm{SM}_i \geq \alpha_{\mathrm{req}}, \qquad \alpha_{\mathrm{req}} = 1 - \frac{1}{\mathrm{SF}_{\mathrm{req}}}.$$

This identity is the key bridge to fuzzy set theory.

3 Safety Margins as Fuzzy Membership Functions

3.1 Dependence on loads

Internal forces are determined by the applied loads $P = (P_1, \ldots, P_n)$ through the equilibrium equations $N_i = N_i(P_1, \ldots, P_n)$. Substituting into Definition 2.2 gives the *load-parameterised safety margin*

$$\mathrm{SM}_i(P_1, \ldots, P_n) = \max\Bigl(0,\ 1 - \frac{|N_i(P_1, \ldots, P_n)|}{A_i\,\sigma_{\max}}\Bigr). \tag{1}$$

3.2 Natural identification with a fuzzy membership function

Definition 3.1 (Fuzzy Set)

Following Zadeh [1], a *fuzzy set* $\tilde{A}$ on a universe X is defined by a membership function $\mu_{\tilde{A}} : X \to [0,1]$, where $\mu_{\tilde{A}}(x)$ is the *degree of membership* of x in $\tilde{A}$.

Proposition 3.2 (Safety Margin Naturally Defines a Membership Function)

The function $\mathrm{SM}_i : \mathbb{R}^n \to [0,1]$ defined by (1) naturally defines a fuzzy membership function on the load space. Under the identification $\mu_i = \mathrm{SM}_i$, the fuzzy set $\tilde{S}_i$ represents load combinations for which member i is *safe to degree* SM_i. The identification is physically meaningful because each α-cut of $\tilde{S}_i$ recovers a classical admissible load domain (Theorem 3.4), and the global minimum of these membership functions reproduces the standard worst-case design criterion (Theorem 4.2).

Remark

No assumptions about uncertainty or randomness are required. The membership function arises from a purely deterministic stress formula; fuzzy algebra provides the appropriate mathematical language to describe its structure.

3.3 Alpha-cuts and admissible load domains

Definition 3.3 (α-Cut)

For $\alpha \in [0,1]$, the α-cut of fuzzy set $\tilde{A}$ on X is defined as in [1, 3] by $[\tilde{A}]_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}$.

Theorem 3.4 (α-Cut = Admissible Load Domain)

The α-cut of the safety fuzzy set $\tilde{S}_i$ is

$$[\tilde{S}_i]_\alpha = \bigl\{P \mid \mathrm{SM}_i(P) \geq \alpha\bigr\} = \bigl\{P \mid |N_i(P)| \leq (1-\alpha)\, A_i\, \sigma_{\max}\bigr\}.$$

This is the admissible load domain for the safety-factor threshold $\mathrm{SF}_{\mathrm{req}} = 1/(1-\alpha)$: the internal force demand is bounded by $(1-\alpha)A_i\sigma_{\max}$, which equals the classical allowable force $A_i\sigma_{\max}$ only when $\alpha = 0$.

Proof. $\mathrm{SM}_i(P) \geq \alpha \Leftrightarrow 1 - |N_i(P)|/(A_i\sigma_{\max}) \geq \alpha \Leftrightarrow |N_i(P)| \leq (1-\alpha)A_i\sigma_{\max}$. □

Remark (Safety Factor as α-Level)

Choosing a required safety factor $\mathrm{SF}_{\mathrm{req}}$ is mathematically equivalent to selecting the α-cut level $\alpha_{\mathrm{req}} = 1 - 1/\mathrm{SF}_{\mathrm{req}}$. For example, $\mathrm{SF}_{\mathrm{req}} = 2 \leftrightarrow \alpha = 0.5$; $\mathrm{SF}_{\mathrm{req}} = 4 \leftrightarrow \alpha = 0.75$. Classical deterministic design operates on a specific α-cut of the underlying fuzzy safety model.

4 Global Safety: Fuzzy Intersection and Worst-Case Analysis

4.1 Series systems and the weakest-link principle

The minimum aggregation rule developed here is appropriate for **nonredundant** (series-system) structures in which failure of any single member constitutes failure of the whole, with no load redistribution. For redundant or ductile structures, alternative aggregation rules would be required.

In the nonredundant setting the structure is safe iff $\mathrm{SM}_i(P_1, P_2) > 0$ for every $i = 1, \dots, m$. The member with the *smallest* margin governs the response (weakest-link principle), so the appropriate global measure is

$$\mathrm{SM}(P_1, P_2) = \min\bigl\{\mathrm{SM}_1(P_1, P_2),\ \dots,\ \mathrm{SM}_m(P_1, P_2)\bigr\}. \tag{2}$$

4.2 Equivalence with fuzzy intersection

Definition 4.1 (Triangular Norms)

A *t-norm* $T : [0,1]^2 \to [0,1]$ satisfies, in the standard sense of fuzzy logic [5], commutativity, associativity, monotonicity, and $T(1, x) = x$. Three standard t-norms are

$$T_G(p, q) = \min(p, q), \quad T_P(p, q) = p \cdot q, \quad T_L(p, q) = \max(0,\, p + q - 1).$$

T_G (Godel) is the largest and unique idempotent t-norm; T_L (Lukasiewicz) is strictly smaller. The *standard fuzzy intersection* uses T_G: $\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$.

Theorem 4.2 (Worst-Case Analysis = Fuzzy Intersection)

The global safety membership function equals the fuzzy intersection (Godel t-norm) of all member-wise safety sets:

$$\mu_{\bigcap_{i=1}^m \tilde{S}_i}(P_1, P_2) \;=\; \mathrm{SM}(P_1, P_2) \;=\; \min_{i=1,\dots,m} \mu_i(P_1, P_2).$$

Classical worst-case structural analysis of a nonredundant structure is a direct application of the Godel fuzzy-intersection formula (2).

Remark (Alternative Aggregation via Other T-norms)

The product t-norm T_P and Lukasiewicz t-norm $T_L(p, q) = \max(0, p + q - 1)$ produce safety aggregation models that differ from the minimum. T_L penalises configurations where one member is near failure even if another has a large margin. Which t-norm best reflects the physical failure behaviour of a given structure is a separate modelling question.

5 Five-Bar Plane Truss: Detailed Example

5.1 Truss geometry, assumptions, and equilibrium

Consider the plane truss (Figure 3) with five bars, a horizontal load P_1 at node A and a vertical load P_2 at node B.

Remark (Modelling Assumptions)

The analysis assumes: (a) linear statics (small deformations); (b) pin-jointed connections (axial forces only, no bending); (c) linear-elastic material up to $\sigma_{\max}$; (d) equal area A and allowable stress $\sigma_{\max}$ for all bars (illustrative; the framework extends to heterogeneous sections).

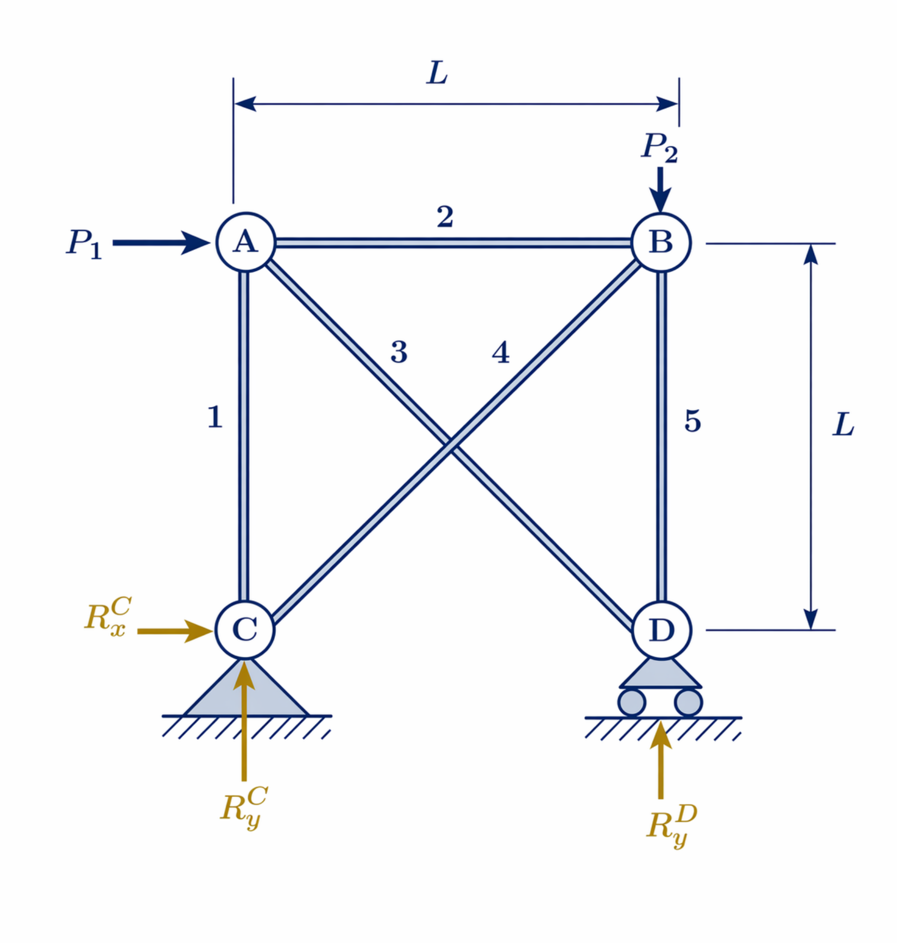


Figure 3: Five-bar plane truss (nodes A–D) with horizontal load P_1 at A and vertical load P_2 at B. Node C is a pin support (R_x^C, R_y^C); node D is a roller (R_y^D). Members 1 and 3 are zero-force members; members 2, 4, and 5 carry the non-trivial internal forces analysed here.

From the method of joints [20]:

$$
\begin{aligned}
&N_1 = 0, && N_2 = -P_1, && N_3 = 0, \\
&N_4 = \sqrt{2}\,P_1, && N_5 = -(P_1 + P_2). && \qquad (3)
\end{aligned}
$$

5.2 Member safety margins

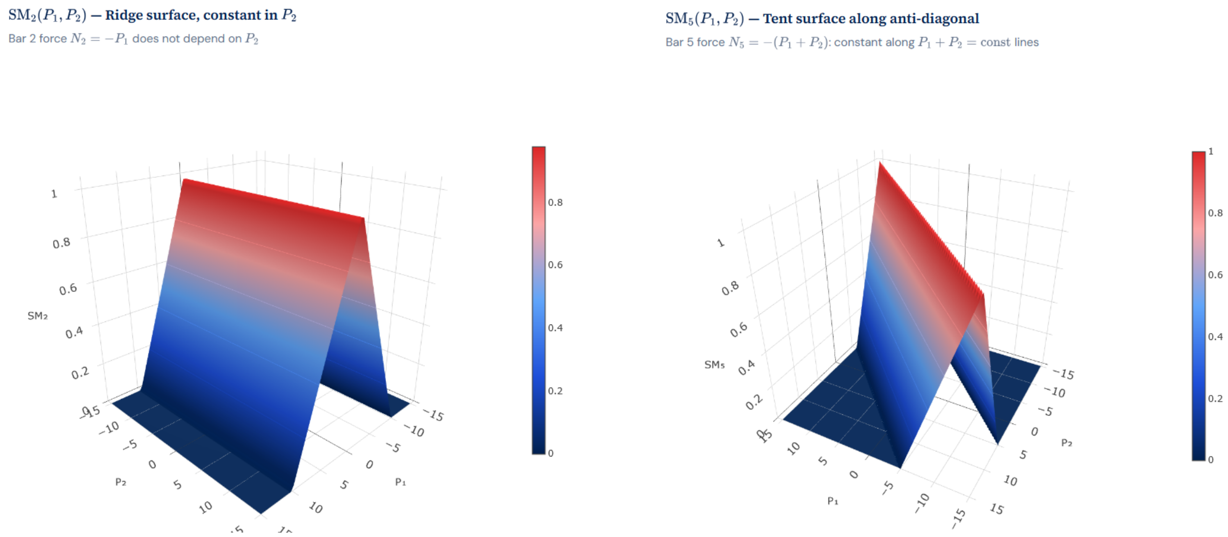
With common area A and allowable stress $\sigma_{\max}$:

$$\mathrm{SM}_2(P_1, P_2) = \max\left(0, 1 - \frac{|P_1|}{A\sigma_{\max}}\right), \tag{4}$$

$$\mathrm{SM}_4(P_1, P_2) = \max\left(0, 1 - \frac{\sqrt{2}|P_1|}{A\sigma_{\max}}\right), \tag{5}$$

$$\mathrm{SM}_5(P_1, P_2) = \max\left(0, 1 - \frac{|P_1+P_2|}{A\sigma_{\max}}\right). \tag{6}$$

Figure 4 shows these surfaces. SM_2 and SM_4 are ridge surfaces (constant in P_2); SM_5 is a tent surface symmetric about $P_1 + P_2 = 0$.



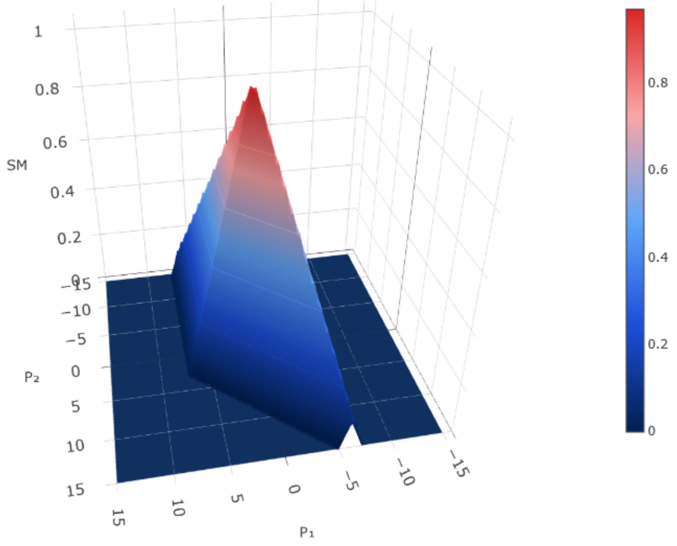
(a) $\mathrm{SM}_2(P_1, P_2)$: function of P_1 only.

(b) $\mathrm{SM}_5(P_1, P_2)$: tent, symmetric about $P_1 + P_2 = 0$.

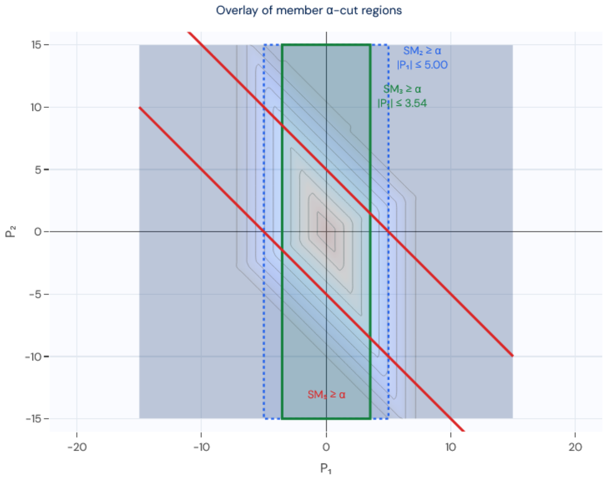
Figure 4: Individual member safety margins over the (P_1, P_2) load plane.

5.3 Global safety margin and its alpha-cut

$$\mathrm{SM}(P_1, P_2) = \min\left\{\mathrm{SM}_2(P_1, P_2),\ \mathrm{SM}_4(P_1, P_2),\ \mathrm{SM}_5(P_1, P_2)\right\}. \tag{7}$$



(a) Global safety margin $\mathrm{SM}(P_1, P_2) = \min_i \mathrm{SM}_i$: pointwise minimum of the individual surfaces.



(b) α-cut $[\tilde{S}]_\alpha$ (top view): parallelogram-shaped admissible region. Every load combination inside satisfies $\mathrm{SM}_i \geq \alpha$.

Figure 5: Global safety margin and its α-cut for the five-bar plane truss.

Applying Theorem 3.4, the α-cut requires:

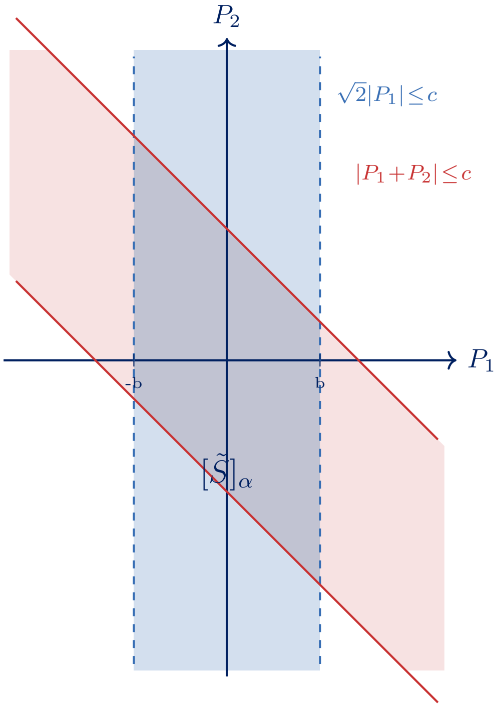
$$|P_1| \leq (1-\alpha)\, A\, \sigma_{\max}, \tag{8}$$

$$\sqrt{2}|P_1| \leq (1-\alpha)\, A\, \sigma_{\max}, \tag{9}$$

$$|P_1 + P_2| \leq (1-\alpha)\, A\, \sigma_{\max}. \tag{10}$$

Constraint (9) (bar 4) is tighter than (8) (bar 2), making (8) redundant. The feasible region is therefore the intersection of bar 4's vertical strip with bar 5's diagonal strip — a *parallelogram* in the (P_1, P_2) plane.

5.4 Schematic of individual and global alpha-cuts



The shaded parallelogram is the α-cut $[\tilde{S}]_\alpha$: the admissible load region where every bar satisfies $\mathrm{SM}_i \geq \alpha$. Its four vertices lie at the intersections of the bar-4 vertical strip ($\sqrt{2}|P_1| = c$, dashed blue) and the bar-5 diagonal boundaries ($P_1 + P_2 = \pm c$, solid red).

6 The Extension Principle, Pullback, and Exact Propagation for Bar 5

6.1 Zadeh's extension principle

Definition 6.1 (Extension Principle)

Following Zadeh's extension principle [2] and standard presentations such as [3, 4], let $f : X_1 \times \cdots \times X_n \to Y$ and let $\tilde{X}_1, \ldots, \tilde{X}_n$ be independent fuzzy inputs. The membership function of the output $\tilde{B} = f(\tilde{X}_1, \ldots, \tilde{X}_n)$ is

$$\mu_{\tilde{B}}(y) = \sup_{\substack{(x_1,\ldots,x_n) \\ f(x_1,\ldots,x_n)=y}} \min\bigl\{\mu_{\tilde{X}_1}(x_1), \ldots, \mu_{\tilde{X}_n}(x_n)\bigr\}.$$

For the present report, the important point is not merely the abstract definition, but whether the classical min-based form reproduces the *specific* safety-margin algebra of bar 5.

6.2 Engineering notation for bar 5 on load space

For bar 5, let

$$N_5 = -(P_1 + P_2), \qquad C := A_5\,\sigma_{\max}.$$

Define the one-load and combined safety profiles by

$$\mu_{\mathrm{SM}_{5,1}}(P_1) = \mathrm{SM}_5(P_1, 0) = \max\left(0, 1 - \frac{|P_1|}{C}\right), \tag{11}$$

$$\mu_{\mathrm{SM}_{5,2}}(P_2) = \mathrm{SM}_5(0, P_2) = \max\left(0, 1 - \frac{|P_2|}{C}\right), \tag{12}$$

$$\mu_{\mathrm{SM}_5}(N_5) = \max\left(0, 1 - \frac{|N_5|}{C}\right). \tag{13}$$

The load-space safety margin is then the pullback

$$\mathrm{SM}_5(P_1, P_2) = \mu_{\mathrm{SM}_5}\big(-(P_1 + P_2)\big).$$

6.3 Why the classical min-based extension gives the wrong profile

If one applies the classical extension principle with the minimum aggregator directly to the one-load safety profiles $\mu_{\mathrm{SM}_{5,1}}$ and $\mu_{\mathrm{SM}_{5,2}}$, one obtains

$$\widehat{\mu}_{\min}(N_5) = \sup_{-(P_1+P_2)=N_5} \min\big(\mu_{\mathrm{SM}_{5,1}}(P_1), \mu_{\mathrm{SM}_{5,2}}(P_2)\big). \tag{14}$$

Theorem 6.2 (Classical min-based extension gives the wrong bar-5 profile)

For bar 5,

$$\widehat{\mu}_{\min}(N_5) = \max\left(0, 1 - \frac{|N_5|}{2C}\right), \tag{15}$$

which is not equal in general to the exact safety profile $\mu_{\mathrm{SM}_5}(N_5) = \max(0, 1 - |N_5|/C)$.

Proof. Using (11)–(12),

$$\min\big(\mu_{\mathrm{SM}_{5,1}}(P_1), \mu_{\mathrm{SM}_{5,2}}(P_2)\big) = \max\left(0, 1 - \frac{\max(|P_1|, |P_2|)}{C}\right).$$

Under the constraint $-(P_1 + P_2) = N_5$, the supremum is attained by balancing the split, namely when $|P_1| = |P_2| = |N_5|/2$ and the signs are chosen so that $P_1 + P_2 = -N_5$. Therefore

$$\widehat{\mu}_{\min}(N_5) = \max\left(0, 1 - \frac{|N_5|}{2C}\right).$$

This differs from (13) unless $N_5 = 0$. ☐

Remark

The failure is structural: the minimum is too weak for additive utilisation sharing. It rewards a balanced decomposition of N_5 into two parts and therefore produces a profile with slope $1/(2C)$ instead of the exact slope $1/C$.

6.4 Exact Łukasiewicz T_L-extension on load space

Let

$$T_L(a,b) := \max(0, a+b-1)$$

denote the strong Łukasiewicz conjunction.

Theorem 6.3 (Exact T_L-extension for bar 5)

For bar 5, the exact force-space safety profile is given by

$$\mu_{\mathrm{SM}_5}(N_5) = \sup_{-(P_1+P_2)=N_5} T_L\big(\mu_{\mathrm{SM}_{5,1}}(P_1), \mu_{\mathrm{SM}_{5,2}}(P_2)\big). \tag{16}$$

Proof. Substituting (11)–(12) into T_L gives

$$T_L\big(\mu_{\mathrm{SM}_{5,1}}(P_1), \mu_{\mathrm{SM}_{5,2}}(P_2)\big) = \max\left(0, 1 - \frac{|P_1| + |P_2|}{C}\right).$$

Under the constraint $-(P_1+P_2) = N_5$, the triangle inequality yields

$$|P_1| + |P_2| \geq |P_1 + P_2| = |N_5|.$$

Hence every admissible pair satisfies

$$T_L\big(\mu_{\mathrm{SM}_{5,1}}(P_1), \mu_{\mathrm{SM}_{5,2}}(P_2)\big) \leq \max\left(0, 1 - \frac{|N_5|}{C}\right).$$

Equality is attained by choosing P_1 and P_2 with the same sign as $-N_5$, so that $|P_1| + |P_2| = |P_1 + P_2| = |N_5|$. Therefore the supremum equals $\mu_{\mathrm{SM}_5}(N_5)$. ☐

Remark

Thus the exact safety profile of bar 5 is obtained not by the classical min-based extension, but by a T_L-extension adapted to additive load sharing. In this sense, the mechanics itself selects the appropriate many-valued connective.

6.5 Constant-sign sectors and consistency with the safety-margin algebra

Proposition 6.4 (Same-sign sector)

If P_1 and P_2 have the same sign, then

$$\mu_{\mathrm{SM}_5}\big(-(P_1+P_2)\big) = T_L\big(\mu_{\mathrm{SM}_{5,1}}(P_1), \mu_{\mathrm{SM}_{5,2}}(P_2)\big) = \max(0, \mathrm{SM}_{5,1} + \mathrm{SM}_{5,2} - 1).$$

This is exactly the strong Łukasiewicz conjunction obtained earlier in the safety-margin algebra.

Proof. If P_1 and P_2 have the same sign, then $|P_1 + P_2| = |P_1| + |P_2|$. Substituting into (13) gives

$$\mu_{\mathrm{SM}_5}\big(-(P_1+P_2)\big) = \max\left(0, 1 - \frac{|P_1| + |P_2|}{C}\right) = T_L\big(\mu_{\mathrm{SM}_{5,1}}(P_1), \mu_{\mathrm{SM}_{5,2}}(P_2)\big).$$

Using (11)–(12) and the notation $\mathrm{SM}_{5,1} = \mathrm{SM}_5(P_1, 0)$, $\mathrm{SM}_{5,2} = \mathrm{SM}_5(0, P_2)$ gives the final form. ☐

Proposition 6.5 (Opposite-sign sector)

If P_1 and P_2 have opposite signs, then

$$\mu_{\mathrm{SM}_5}\big(-(P_1+P_2)\big) = 1 - \big|\mathrm{SM}_{5,1} - \mathrm{SM}_{5,2}\big|.$$

Thus the exact load-space profile remains (13), but when it is re-expressed in terms of the one-load margins on this sector it becomes the Łukasiewicz equivalence.

Proof. If P_1 and P_2 have opposite signs, then

$$|P_1+P_2| = \big||P_1| - |P_2|\big|.$$

Using (13),

$$\mu_{\mathrm{SM}_5}\big(-(P_1+P_2)\big) = \max\left(0, 1 - \frac{\big||P_1| - |P_2|\big|}{C}\right).$$

Because $|P_1| = (1-\mathrm{SM}_{5,1})C$ and $|P_2| = (1-\mathrm{SM}_{5,2})C$ on the unsaturated branch, this becomes

$$\mu_{\mathrm{SM}_5}\big(-(P_1+P_2)\big) = 1 - \big|\mathrm{SM}_{5,1} - \mathrm{SM}_{5,2}\big|.$$

□

6.6 Pullback of a safety criterion

The function $\mu_{\mathrm{SM}_5}(N_5)$ in (13) is first defined on the force axis of bar 5. Composing it with the equilibrium map $(P_1, P_2) \mapsto N_5 = -(P_1+P_2)$ gives the corresponding load-space safety margin:

$$\mathrm{SM}_5(P_1, P_2) = \mu_{\mathrm{SM}_5}\big(-(P_1+P_2)\big). \tag{17}$$

This is a pullback, not a min-based forward extension. It simply re-expresses the force-space safety criterion in load-space coordinates.

Theorem 6.6 (Pullback identity)

For any $\alpha \in [0,1]$:

$$\big\{(P_1,P_2) \mid \mathrm{SM}_5(P_1,P_2) \ge \alpha\big\} = \big\{(P_1,P_2) \mid \mu_{\mathrm{SM}_5}(-(P_1+P_2)) \ge \alpha\big\}.$$

Proof. This is immediate from (17). Both descriptions reduce to $|P_1+P_2| \le (1-\alpha)A_5\sigma_{\max}$. □

7 The Dependency Problem and Consistency Constraints

7.1 Independence assumption and its failure

One might form the global safety margin by minimising over individual force-space representations:

$$\mathrm{SM}^*(N_1,\dots,N_5) = \min_i \mathrm{SM}_i^*(N_i). \quad \text{(incorrect)} \tag{18}$$

However, the N_i are *not independent*: they all arise from the same loads (P_1, P_2) via equilibrium. Treating them as independent inflates the solution set.

7.2 The dependency problem in interval arithmetic

In interval arithmetic this is the *dependency problem* [12, 13, 14]: the same variable appearing in multiple terms is treated as independent, causing overestimation. Canonical example: $x \in [0,1] \Rightarrow x - x = 0$, but $[0,1] - [0,1] = [-1,1]$. The same inflation occurs when P_1 appears in both $N_4 = \sqrt{2}P_1$ and $N_5 = -(P_1 + P_2)$.

Theorem 7.1 (Correct Global Safety via Equilibrium Constraints)

The correct global safety margin enforces the equilibrium:

$$\mathrm{SM}(P_1, P_2) = \min\big\{\mathrm{SM}_2^*(N_2),\ \mathrm{SM}_4^*(N_4),\ \mathrm{SM}_5^*(N_5)\big\} \quad \text{with} \quad \begin{cases} N_2 = -P_1, \\ N_4 = \sqrt{2}\,P_1, \\ N_5 = -(P_1 + P_2). \end{cases}$$

The equilibrium constraints are the compatibility conditions that resolve the dependency problem in fuzzy arithmetic.

8 Safety Margin as a Multivalued Degree of Truth

8.1 Background: many-valued logics

Classical logic assigns truth values in $\{0,1\}$. Łukasiewicz [7] and Post [8] introduced many-valued logics with truth values in $[0,1]$. Two infinite-valued systems are particularly relevant:

Godel logic uses the minimum t-norm $T_G(p,q) = \min(p,q)$ for conjunction and the maximum for disjunction.

Łukasiewicz logic uses its own t-norm $T_L(p,q) = \max(0, p+q-1)$ for the strong conjunction, alongside negation $\neg p = 1 - p$ and disjunction $\min(1, p+q)$ [10, 11].

The safety aggregation rule in this paper — taking the global minimum over all bars — corresponds to the **Godel conjunction** T_G, not the Lukasiewicz t-norm.

Remark (Historical Note)

Łukasiewicz introduced three-valued logic in 1920 and later generalised it [7, 10]. Godel introduced his minimum-based logic in 1932. The connection between $[0,1]$-valued logic and fuzzy sets was made explicit by Goguen [9].

8.2 The safety margin as a degree of truth

Consider the proposition $\mathcal{P}(P_1, P_2)$: “The structure is safe under loads P_1, P_2.” In classical logic, this is true or false. Here it admits a graded truth value:

Theorem 8.1 (Safety Margin as Graded Truth)

Define the degree of truth of $\mathcal{P}$ by

$$\mu_{\mathrm{SM}}(P_1, P_2) \;=\; \mathrm{SM}(P_1, P_2) \;=\; \min_{i=1,\dots,m} \mathrm{SM}_i(P_1, P_2). \tag{19}$$

Then:

1. $\mu_{\mathrm{SM}} : \mathbb{R}^2 \to [0,1]$, as required by the unit-interval truth scale.
2. $\mu_{\mathrm{SM}} = 1$ iff all bars are unloaded (vacuously safe).
3. $\mu_{\mathrm{SM}} = 0$ iff at least one bar reaches its allowable stress.
4. $0 < \mu_{\mathrm{SM}} < 1$ represents partial safety: gradual transition from safe to unsafe.
5. The minimum in (19) is the *Godel/lattice conjunction* on $[0,1]$. The Lukasiewicz t-norm $T_L(p,q) = \max(0, p+q-1)$ and product t-norm lead to different, stricter safety models.
6. The α-cut $[\tilde{S}]_\alpha$ is the set of load combinations for which $\mathcal{P}$ is “true to degree $\geq \alpha$”, corresponding to safety-factor threshold $\mathrm{SF}_{\mathrm{req}} = 1/(1-\alpha)$.

Remark (Physical Interpretability)

Formula (19) provides a *fully interpretable* membership function: every parameter has a direct physical meaning (A_i: cross-section; $\sigma_{\max}$: material constant; N_i: equilibrium force; SM_i: normalised stress reserve). Unlike typical fuzzy models where the membership shape is chosen heuristically, here it is *derived* from the structural physics.

8.3 Summary of equivalences

Table 1: Mathematical equivalences between structural mechanics, fuzzy algebra, and many-valued logic.

Structural Mechanics	**Fuzzy Algebra**	**Many-Valued Logic**
Safety margin $\mathrm{SM}_i(P) \in [0,1]$	Membership function $\mu_i(P)$	Degree of truth "bar i safe"
$\mathrm{SM}_i(P) \geq \alpha$	α-cut $[\tilde{S}_i]_\alpha$	Proposition true to degree $\geq \alpha$
Safety factor $\mathrm{SF}_{\mathrm{req}}$	α-level $1 - 1/\mathrm{SF}_{\mathrm{req}}$	Acceptance threshold
Global worst-case minimum	Fuzzy intersection (Godel T_G)	Godel lattice conjunction
Exact force-space profile of bar 5	$\mu_{\mathrm{SM}_5}(N_5) = \max(0, 1 - \lvert N_5\rvert/C)$	Graded truth on the force axis
Classical min-extension for bar 5	$\widehat{\mu}_{\min}(N_5) = \max(0, 1 - \lvert N_5\rvert/(2C))$	Incorrect profile for additive utilisation sharing
Exact T_L-extension on load space	$\mu_{\mathrm{SM}_5}(N_5) = \sup_{-(P_1+P_2)=N_5} T_L(\mu_{\mathrm{SM}_{5,1}}(P_1), \mu_{\mathrm{SM}_{5,2}}(P_2))$	Exact graded image of additive superposition
Safety criterion in force space	Pullback / pre-image	Composition with truth-valued map
Dependency / equilibrium constraint	Consistent argument in extension	Shared variable in compound formula
Admissible force domain (α-level)	α-cut of safety fuzzy set	Set of sufficiently true instances
Same-sign superposition in one bar	$\mathrm{SM}_5 = \mathrm{SM}_{5,1} \otimes_{\mathrm{L}} \mathrm{SM}_{5,2}$	Strong Łukasiewicz conjunction
Opposite-sign cancellation in one bar	$\mathrm{SM}_5 = \mathrm{SM}_{5,1} \Leftrightarrow_{\mathrm{L}} \mathrm{SM}_{5,2}$	Łukasiewicz equivalence
Residual capacity under one load	Residuum $a \Rightarrow_{\mathrm{L}} b$	Łukasiewicz implication

9 Single-Bar Superposition Algebra for Bar 5

9.1 Separate one-load margins for bar 5

For bar 5 of the truss,

$$N_5 = -(P_1 + P_2), \qquad C := A_5\,\sigma_{\max},$$

so the combined safety margin is

$$\mathrm{SM}_5(P_1, P_2) = \max\left(0,\ 1 - \frac{|P_1 + P_2|}{C}\right). \tag{20}$$

To describe the effect of each load on *the same bar*, define the individual one-load margins

$$\mathrm{SM}_{5,1} := \mathrm{SM}_5(P_1, 0) = \max\left(0,\ 1 - \frac{|P_1|}{C}\right), \qquad \mathrm{SM}_{5,2} := \mathrm{SM}_5(0, P_2) = \max\left(0,\ 1 - \frac{|P_2|}{C}\right). \tag{21}$$

These are not the margins of bars 1 and 2. They are the margins of *bar 5* under the separate actions of P_1 and P_2.

When $\mathrm{SM}_{5,1} > 0$ and $\mathrm{SM}_{5,2} > 0$, the safety map is on its unsaturated affine branch, so the load magnitudes can be recovered from these margins:

$$|P_1| = (1 - \mathrm{SM}_{5,1})C, \qquad |P_2| = (1 - \mathrm{SM}_{5,2})C. \tag{22}$$

Writing

$$P_k = \varepsilon_k |P_k|, \qquad \varepsilon_k := \operatorname{sgn}(P_k) \in \{-1, +1\},$$

we obtain the exact sign-resolved representation

$$P_1 = \varepsilon_1(1 - \mathrm{SM}_{5,1})C, \qquad P_2 = \varepsilon_2(1 - \mathrm{SM}_{5,2})C. \tag{23}$$

Theorem 9.1 (Exact composition law for bar 5)

Assume $\mathrm{SM}_{5,1} > 0$ and $\mathrm{SM}_{5,2} > 0$. Then the combined margin of bar 5 is given exactly by

$$\boxed{\mathrm{SM}_5(P_1, P_2) = \max\big(0,\ 1 - \big|\varepsilon_1(1 - \mathrm{SM}_{5,1}) + \varepsilon_2(1 - \mathrm{SM}_{5,2})\big|\big).} \tag{24}$$

Proof. Substituting (23) into (20) gives

$$\mathrm{SM}_5(P_1, P_2) = \max\left(0,\ 1 - \frac{|P_1 + P_2|}{C}\right) = \max\big(0,\ 1 - \big|\varepsilon_1(1 - \mathrm{SM}_{5,1}) + \varepsilon_2(1 - \mathrm{SM}_{5,2})\big|\big).$$

□

Remark (Scope of exactness)

The composition law (24) is exact precisely on the unsaturated branch, where the map $a \mapsto \max(0, 1 - a)$ is invertible. If, for example, $\mathrm{SM}_{5,1} = 0$, then one only knows $|P_1| \geq C$ and cannot reconstruct the exact magnitude of P_1 from the margin alone. Thus no exact binary operation on $\mathrm{SM}_{5,1}$ and $\mathrm{SM}_{5,2}$ can recover $\mathrm{SM}_5(P_1, P_2)$ beyond the affine branch without additional information.

9.2 Constant-sign sectors and induced Lukasiewicz operations

Theorem 9.2 (Co-directional loads: strong Lukasiewicz conjunction)

If $\varepsilon_1 = \varepsilon_2$, then

$$\boxed{\mathrm{SM}_5(P_1, P_2) = \mathrm{SM}_{5,1} \otimes_{\mathrm{L}} \mathrm{SM}_{5,2} := \max(0,\ \mathrm{SM}_{5,1} + \mathrm{SM}_{5,2} - 1).} \tag{25}$$

Proof. When $\varepsilon_1 = \varepsilon_2$, the magnitudes add:

$$|P_1 + P_2| = |P_1| + |P_2| = (1 - \mathrm{SM}_{5,1})C + (1 - \mathrm{SM}_{5,2})C.$$

Substituting into (20) yields

$$\mathrm{SM}_5(P_1, P_2) = \max\left(0,\ 1 - \frac{|P_1 + P_2|}{C}\right) = \max(0,\ \mathrm{SM}_{5,1} + \mathrm{SM}_{5,2} - 1).$$

$\square$

Remark (Why the strong Lukasiewicz conjunction appears)

Force superposition is additive, whereas the safety map $\mathrm{SM} = 1 - |\cdot|/C$ is affine on the unsaturated branch. Composing an additive law with an affine safety map therefore yields the additive t-norm $\max(0, a + b - 1)$. Thus the strong Łukasiewicz conjunction is not imposed externally; it is induced by the mechanics of co-directional loading in bar 5.

Proposition 9.3 (Dual form on utilisations)

Define the utilisation fractions

$$r_{5,1} := 1 - \mathrm{SM}_{5,1} = \frac{|P_1|}{C}, \qquad r_{5,2} := 1 - \mathrm{SM}_{5,2} = \frac{|P_2|}{C}, \qquad r_5 := 1 - \mathrm{SM}_5(P_1, P_2).$$

Under co-directional loading,

$$r_5 = \min(1,\ r_{5,1} + r_{5,2}) =: r_{5,1} \oplus_{\mathrm{L}} r_{5,2}. \tag{26}$$

Hence safety margins combine by the strong conjunction $\otimes_{\mathrm{L}}$, whereas utilisations combine by the strong disjunction $\oplus_{\mathrm{L}}$. The two are De Morgan dual under the negation $a \mapsto 1 - a$.

Theorem 9.4 (Opposite signs: Lukasiewicz equivalence)

If $\varepsilon_1 = -\varepsilon_2$, then

$$\boxed{\mathrm{SM}_5(P_1, P_2) = 1 - |\mathrm{SM}_{5,1} - \mathrm{SM}_{5,2}| =: \mathrm{SM}_{5,1} \Leftrightarrow_{\mathrm{L}} \mathrm{SM}_{5,2}.} \tag{27}$$

Proof. When $\varepsilon_1 = -\varepsilon_2$, the two load effects partially cancel, so

$$|P_1 + P_2| = \big||P_1| - |P_2|\big| = |(1 - \mathrm{SM}_{5,1}) - (1 - \mathrm{SM}_{5,2})|\, C = |\mathrm{SM}_{5,1} - \mathrm{SM}_{5,2}|\, C.$$

Substituting into (20) gives

$$\mathrm{SM}_5(P_1, P_2) = \max(0,\ 1 - |\mathrm{SM}_{5,1} - \mathrm{SM}_{5,2}|) = 1 - |\mathrm{SM}_{5,1} - \mathrm{SM}_{5,2}|,$$

because $|\mathrm{SM}_{5,1} - \mathrm{SM}_{5,2}| \leq 1$. $\square$

Remark (Mechanical meaning of the equivalence)

Equation (27) equals 1 exactly when $\mathrm{SM}_{5,1} = \mathrm{SM}_{5,2}$. Thus equal-magnitude opposite loads cancel perfectly and restore full safety in bar 5. The Łukasiewicz equivalence therefore measures the degree to which the two one-load states balance each other.

Proposition 9.5 (Implication and residual capacity)

Fix a target combined margin $b \in [0,1]$ in the co-directional case. Then

$$\mathrm{SM}_{5,1} \otimes_{\mathrm{L}} \mathrm{SM}_{5,2} \geq b \quad \Longleftrightarrow \quad \mathrm{SM}_{5,2} \geq \mathrm{SM}_{5,1} \Rightarrow_{\mathrm{L}} b := \min(1,\, 1 - \mathrm{SM}_{5,1} + b). \tag{28}$$

Hence the residuum of the strong conjunction quantifies the minimum margin that the second load must preserve in order to guarantee the required final safety level b.

Proof. By (25),

$$\mathrm{SM}_{5,1} \otimes_{\mathrm{L}} \mathrm{SM}_{5,2} \geq b \iff \max(0,\, \mathrm{SM}_{5,1} + \mathrm{SM}_{5,2} - 1) \geq b.$$

Since $b \geq 0$, this is equivalent to

$$\mathrm{SM}_{5,1} + \mathrm{SM}_{5,2} - 1 \geq b \iff \mathrm{SM}_{5,2} \geq 1 - \mathrm{SM}_{5,1} + b.$$

Because margins lie in $[0,1]$, the smallest admissible lower bound is $\min(1, 1 - \mathrm{SM}_{5,1} + b)$. ☐

9.3 Why no single sign-blind connective exists on margin space

The exact deterministic law for bar 5 depends not only on the two one-load margins $\mathrm{SM}_{5,1}$ and $\mathrm{SM}_{5,2}$, but also on the sign sector of (P_1, P_2). If one tries to suppress that information and search for a single binary connective on margin values alone, the mechanics becomes non-representable.

Theorem 9.6 (No single sign-blind connective recovers all cases)

There is no single function $H : [0,1]^2 \to [0,1]$ such that

$$\mathrm{SM}_5(P_1, P_2) = H(\mathrm{SM}_{5,1}, \mathrm{SM}_{5,2})$$

for all load pairs (P_1, P_2) on the unsaturated branch. Any exact composition law must also retain the sign sector, or some information equivalent to it.

Proof. Choose load pairs with the same one-load margins but different sign patterns. For example, let $|P_1| = |P_2| = C/2$, so that $\mathrm{SM}_{5,1} = \mathrm{SM}_{5,2} = 1/2$.

If P_1 and P_2 have the same sign, then by Theorem 9.2,

$$\mathrm{SM}_5(P_1, P_2) = \max(0, 1/2 + 1/2 - 1) = 0.$$

If they have opposite signs, then by Theorem 9.4,

$$\mathrm{SM}_5(P_1, P_2) = 1 - |1/2 - 1/2| = 1.$$

Thus the same pair $(\mathrm{SM}_{5,1}, \mathrm{SM}_{5,2}) = (1/2, 1/2)$ would have to be mapped simultaneously to both 0 and 1, which is impossible for a single-valued function H. ☐

Remark

Section 6 shows how the exact force-space safety profile of bar 5 is still recovered on load space: the classical min-based extension fails, but the Łukasiewicz T_L-extension reproduces the correct profile exactly. Once the sign sector is fixed, the resulting formulas reduce to the constant-sign connectives derived above.

Remark (Two logical layers)

The updated theory therefore has two distinct logical levels.

- **Across bars in a nonredundant structure:** global safety is the Godel minimum, i.e. weak conjunction.
- **Within one bar under superposition:** bar 5 carries an induced Łukasiewicz-type algebra on the one-load margins $\mathrm{SM}_{5,1}$ and $\mathrm{SM}_{5,2}$, with the strong conjunction, equivalence, and residuum appearing on different sign sectors or in residual-capacity questions.

These layers describe different mechanical questions and should not be identified with each other.

10 Broader implications for logic, classification, and AI

The equivalence developed in this report suggests a broader mathematical programme. In the present structural example, the algebra of safety margins is derived entirely from mechanics: each quantity has a clear engineering meaning, and each operation corresponds to a physically meaningful transformation of loads, forces, and capacity reserves [19, 20]. The central observation is that this deterministic algebra can be rewritten exactly in the language of fuzzy sets [1] and many-valued logic [7] without altering the underlying mathematics. Membership functions and truth degrees are therefore not heuristic additions here; they are an alternative notation for a fully interpretable engineering theory [6].

A similar approach may apply wherever degrees of truth, admissibility, or performance can be described and interpreted precisely [16]. Real phenomena rarely divide into perfectly crisp classes: gradual transitions, partial satisfaction of conditions, and mixed states are ubiquitous in engineering and decision-making [4, 3]. For such phenomena, binary logic is an idealization that can obscure structure the mathematics actually contains [9]. An algebra of graded quantities—in which logical operations arise from physical laws or data rather than from heuristic choices—can provide both formal precision and direct interpretability, and may yield new mathematical proofs, computational algorithms, and reasoning procedures in areas where graded concepts arise naturally [10, 5].

This perspective is especially relevant for classification and artificial intelligence. Many real phenomena are better described by continuous stages or partial membership than by sharply separated classes [11]. Extending the present framework [15], one may derive exact formulas for reasoning with continuously graded concepts—formulas in which each intermediate step has a precise meaning, each truth degree is explicitly defined, and the final result remains transparent and verifiable. The key message is that fuzziness need not imply vagueness: in favourable cases, degrees of truth can be defined exactly, interpreted directly, and manipulated rigorously, potentially leading to AI systems that are not only more flexible but also more interpretable and safer to deploy [2].

11 Conclusion

We have shown that classical structural safety analysis is a concrete realisation of fuzzy algebra and many-valued logic in the nonredundant (series-system) setting:

1. The **safety margin** $\mathrm{SM}_i(P_1, P_2)$ naturally defines a fuzzy membership function on the load space, mapping every load combination to a degree of safety in $[0, 1]$.
2. The **admissible load domain** for safety-factor threshold $\mathrm{SF}_{\mathrm{req}}$ is exactly the α-cut at level $\alpha = 1 - 1/\mathrm{SF}_{\mathrm{req}}$. The force bound is $(1-\alpha)A_i\sigma_{\max}$, not the classical allowable $A_i\sigma_{\max}$ (which corresponds to $\alpha = 0$ only).
3. **Across bars**, global worst-case analysis is the standard fuzzy intersection formula using the Godel/lattice conjunction $\min(p, q)$.
4. **Within one bar**, superposition of two loads induces a different algebra on the one-load margins $\mathrm{SM}_{5,1} = \mathrm{SM}_5(P_1, 0)$ and $\mathrm{SM}_{5,2} = \mathrm{SM}_5(0, P_2)$. For bar 5 on the unsaturated branch, co-directional loading gives the strong Łukasiewicz conjunction $\mathrm{SM}_{5,1} \otimes_{\mathrm{L}} \mathrm{SM}_{5,2} = \max(0, \mathrm{SM}_{5,1} + \mathrm{SM}_{5,2} - 1)$, whereas opposite-sign loading gives the Łukasiewicz equivalence $\mathrm{SM}_{5,1} \Leftrightarrow_{\mathrm{L}} \mathrm{SM}_{5,2} = 1 - |\mathrm{SM}_{5,1} - \mathrm{SM}_{5,2}|$.
5. The **residuum** $\mathrm{SM}_{5,1} \Rightarrow_{\mathrm{L}} b = \min(1, 1 - \mathrm{SM}_{5,1} + b)$ has a direct engineering interpretation: it is the minimum remaining margin required from the second load in order to guarantee the target combined safety level b.
6. **No single sign-blind connective** on the pair $(\mathrm{SM}_{5,1}, \mathrm{SM}_{5,2})$ can recover all deterministic superposition cases for bar 5. The missing information is the sign sector of the loads.
7. **Load propagation through equilibrium** (when loads are fuzzy) is the forward extension principle on load space. Propagation of fuzzy one-load margins for bar 5 instead requires a **generalised extension principle on augmented margin space**, where the mechanically induced map G_σ is coupled with the sign-sector variable.
8. Re-expressing a force-space safety criterion on the load space is a **pullback** (function composition), distinct from both the forward load-space extension and the induced margin-space extension.
9. The **dependency problem** in fuzzy/interval arithmetic corresponds to violating equilibrium constraints; enforcing them recovers the exact classical result.
10. The function $\mu = \mathrm{SM}$ provides a *physically derived, fully interpretable* membership function and degree of truth, without any arbitrary parameter choices.

The updated framework therefore has two complementary logical layers: a **weak** Godel algebra across members of a nonredundant structure, and a **strong** Łukasiewicz-type algebra within a single member under load superposition. Both arise from mechanics rather than from expert elicitation. This makes the connection between structural safety, fuzzy algebra, generalised extension principles, and many-valued logic substantially richer than the global minimum rule alone.

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